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THE LUCKY NUMBER THEOREM

D. Hawkins and W. E. Briggs

The lucky numbers of Ulam resemble prime numbers in their apparent distribution among the natural numbers and with respect to the kind of sieve that generates them [1]. It is therefore of interest to investigate their properties, bearing in mind the analogies to prime number theory.

The lucky numbers are defined by the following sieve. If $S_n$ is an infinite sequence of natural numbers $t_{n,m}$ ($m = 1, 2, 3, \ldots$), one obtains $S_{n+1}$, for $n > 1$, from $S_n$ by removing every $t_{n,m}$ for which $t_{n,n}$ divides $m$. $S_2$ is the sequence 2, 3, 5, 7, 9, \ldots of the number 2 followed by the odd integers in increasing order of magnitude. $S_1$ is the sequence of natural numbers. The sequence of lucky numbers is $S = \lim_{n \to \infty} S_n$, that is, 2, 3, 7, 9, 13, 15, 21, \ldots. This definition differs trivially from Ulam's.

Two properties are basic for the investigation of asymptotic properties of lucky numbers. By the definition if $s_n$ represents the $m$-th lucky number, then

$$s_m = t_{n,m} \quad \text{for all } m < s_n.$$  

This follows from the fact that $t_{n,s(n)}$ is the first number that will be removed from $S_n$ in forming $S_{n+1}$, and is at the same time less than any number to be removed later on. Also by the definition, if $R(n, x)$ is the number of numbers not greater than $x$ in $S_n$, and $[x]$ denotes the greatest integer in $x$, then

$$R(n, x) = R(n-1, x) \left[ \frac{R(n-1, x)}{s_{n-1}} \right] n \geq 2.$$  

This fundamental recurrence relation has the following solution, in which $\lfloor x \rfloor$ denotes the fractional part of $x$ and $\sigma_n = (1 - 1/2)(1 - 1/3)(1 - 1/7) \ldots (1 - 1/s_{n-1})$,

$$R(n, x) = \lfloor x \rfloor \sigma_n + \sum_{i=2}^{n} \sigma_i \left\{ \frac{R(i-1, x)}{s_{i-1}} \right\}, \ n \geq 2,$$

Clearly

$$277.$$
The properties of $S$ now develop by a series of stages. The first is to find bounds for $s_n$. If one puts $R(n, s_{n+r}) = n+r$, which by (1) one may do for $0 \leq r < s_n - n$, then

\begin{equation}
\frac{s_{n+1}}{s_n} = 1 - \frac{1}{n} \leq 1 - \frac{s_n}{n}, \quad n \geq 2.
\end{equation}

Putting $\rho_n = 1/\sigma_n$,

\begin{equation}
\rho_{n+1} - \rho_n \geq \frac{\rho_{n+1}}{n\rho_n} > \frac{1}{n}, \quad n \geq 2.
\end{equation}

Summing from 2 to $n-1$, one obtains

\begin{equation}
\rho_n - \rho_2 > \sum_{t=2}^{n-1} \frac{1}{t}
\end{equation}

which implies $\rho_n > \log n$, or

\begin{equation}
\sigma_n < \frac{1}{\log n}, \quad n \geq 2.
\end{equation}

In $R(n, s_{n+r}) = n+r$ ($0 \leq r < s_n - n$), one may set $r = n$ and $r = n-1$, since $s_n > 2n$ for $n>2$, which gives, by (3),

\begin{equation}
2n = s_{2n} + E(n, s_{2n})
\end{equation}

\begin{equation}
2n - 1 = s_{2n-1} + E(n, s_{2n-1}).
\end{equation}

Hence, by (4) and (6), for $n>2$,

\begin{equation}
\frac{s_{2n}}{\log n} > n \log n
\end{equation}

\begin{equation}
\frac{s_{2n-1}}{(n-1) \log n} < (n-1) \log n.
\end{equation}

It is now possible to show that the remainder term $E(n, x)$ is $o(n)$ when $x = s_n$. For from (1) and (3),

\begin{equation}
R(n, s_n) = s_n + E(n, s_n) = n.
\end{equation}

Let $\alpha(n)$ be the integer defined by $s_{\alpha(n)} \geq n$ and $s_{\alpha(n)-1} < n$. By (7)
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\(\sigma(n) < \frac{3n}{\log n} \quad n > n_0.\)

Now split the sum \(E(n, s_n)\) into two parts \(E_1\) and \(E_2\), where

\[
E_1 = \sum_{i=2}^{3n/\log n} \frac{\sigma_n}{\sigma_i} \left\{ \frac{R(i-1, s_n)}{s_{i-1}} \right\},
\]

\[
E_2 = \sum_{i=3n/\log n}^{n} \frac{\sigma_n}{\sigma_i} \left\{ \frac{R(i-1, s_n)}{s_{i-1}} \right\}.
\]

Clearly

\[
E_1 = 0\left(\frac{n}{\log n}\right).
\]

In \(E_2\), because of (1), put all \(R(i-1, s_n) = R(\sigma(n), s_n) = n\), so that by (7)

\[
E_2 = 0\left(\sum_{i=3n/\log n}^{n} \frac{n}{i \log i} \right) = 0\left(\frac{n \log \log n}{\log n}\right).
\]

It is now possible to write for \(n \geq 2,

\[
\frac{\sigma_{n+1}}{\sigma_n} = 1 - \frac{1}{s_n} = 1 - \frac{\sigma_n}{n + o(n)}.
\]

Again using the substitution \(\rho_n = 1/\sigma_n\), one obtains

\[
\rho_{n+1} - \rho_n = \frac{\rho_{n+1}}{n \rho_n} + o\left(\frac{1}{n}\right) \frac{\rho_{n+1}}{\rho_n}.
\]

By summation \(\rho_n = \log n + o(\log n)\) and, therefore,

\[
\sigma_n \sim \frac{1}{\log n},
\]

and from this, (1), and (3)

\[
s_n = \frac{n + o(n)}{\sigma_n} \sim n \log n.
\]

(15) is the analogue of the Merten's theorem for prime numbers, and
(16) is the analogue of the prime number theorem.

These results, especially (16), confirm a conjecture of one of the authors based on stochastic arguments [2]. They support the observation that the asymptotic distribution of prime numbers is not, except in details, a consequence of their primality, but characteristic of a wide class
of sieve-generated sequences, of which the lucky numbers are an example. By using the results recursively in (14), S. Chowla has shown that the asymptotic value of (15) can be improved to

\begin{equation}
 s_n = n \log n + \frac{n}{2} (\log \log n)^2 + o(n(\log \log n)^2).
\end{equation}

Since the corresponding result for prime numbers (where \( p_n \) is the \( n \)-th prime number) is

\begin{equation}
 p_n = n \log n + n \log \log n + o(n \log \log n),
\end{equation}

it follows, with only a finite number of exceptions, that \( s_n > p_n \). With necessary calculations, this presumably will confirm Ulam's conjecture \( s_n > p_n \) for all \( n \).

REFERENCES
