\[ \{ g((a_1, \ldots, a_n)) \mid f((a_1, \ldots, a_n)) = i \} = \{ 0, 1, \ldots, m^{n-1} - 1 \} \] when 
\[ i = 0, \ldots, m-1. \] Note that there is at least one Latin chessboard, e.g.: 
\[ f((a_1, \ldots, a_n)) \equiv a_1 + a_2 + \cdots + a_n \pmod{m}. \] It can be easily shown that we minimize connectedness parallel to the \( i \)-axis by distributing the points of \( S \) as evenly as possible among the \( m^{n-1} \) pencils parallel to the \( i \)-axis. The assignments 
\[ h((a_1, \ldots, a_n)) = m^{n-1}f((a_1, \ldots, a_n)) + g((a_1, \ldots, a_n)) \] are exactly those assignments which do this for all \( i = 1, \ldots, n. \)

The maximum value attained is \( (1/6)n(m^2-1)m^{2n-1} \). Thus the minimum assignment is \( 1/n \)th the average assignment, asymptotically as \( m \) or \( n \) gets large, whereas the average assignment is \( (m/m+1) \)th of the maximum assignment as \( m \) gets large.

Reference


This paper was one of two winners of the first E. T. Bell prize for undergraduate research in mathematics at the California Institute of Technology. The problem arose at the Jet Propulsion Laboratory, which the Institute operates with support from the National Aeronautics and Space Administration.

Editorial Note. In this MONTHLY 70 (1963) 706–711, A. A. Blank asked whether \( \pi/8 \) may be the minimal area of a star-shaped domain within which a unit segment can be turned through 360°. C. S. Ogilvy has called attention to a demonstration by R. J. Walker (Pi Mu Epsilon Journal, 1 (1952) 275) that a unit segment can be turned through 360° in a five-pointed star with area approximately three quarters of \( \pi/8 \).

MATHEMATICAL NOTES

Edited by J. H. Curtiss, University of Miami

Material for this department should be sent to J. H. Curtiss,
University of Miami, Coral Gables 46, Florida

A VISUAL DISPLAY OF SOME PROPERTIES OF THE DISTRIBUTION OF PRIMES

M. L. Stein, S. M. Ulam, and M. B. Wells, University of California,
Los Alamos Scientific Laboratory, Los Alamos, New Mexico

Suppose we number the lattice points in the plane by a single sequence, e.g. Fig. 1 by starting at \((0, 0)\) and proceeding counterclockwise in a spiral so that 
\[ (0, 0) \rightarrow 1, (1, 0) \rightarrow 2, (1, 1) \rightarrow 3, (0, 1) \rightarrow 4, (-1, 1) \rightarrow 5, (-1, 0) \rightarrow 6, (-1, -1) \rightarrow 7, (0, -1) \rightarrow 8, (1, -1) \rightarrow 9, (2, -1) \rightarrow 10, (2, 0) \rightarrow 11, (2, 1) \rightarrow 12, (2, 2) \rightarrow 13, \text{ etc.} \]
Consider the set $P$ of those lattice points whose single index becomes a prime. Under our correspondence, points of $P$ located on straight lines have indices which ultimately consist of values of a quadratic form. This is easily seen because the third differences between neighboring points on a straight line are 0 and after a finite number of indices which vary linearly have been passed, the progression becomes truly quadratic.

The set $P$ appears to exhibit a strongly nonrandom appearance (i.e. a different appearance from randomly chosen sets whose densities are like those of primes; that is, asymptotically $\log n/n$). This is due, of course, to the fact that some lines corresponding to quadratic forms which are factorable are devoid of primes; some other quadratic forms are rich in primes. A glance at a picture showing the set $P$ reveals many such lines. It is a property of the visual brain which allows one to discover such lines at once and also notice many other peculiarities of distribution of points in two dimensions. In a visualization of a one-dimensional sequence this is not so much the case. (Perhaps an acoustic interpretation would be more suggestive?)

In addition to the well-known Euler form: $y^2+y+41$, one could observe instantly many other prime-rich forms. One line rather prominent in Fig. 1 in the lower half of the picture has numbers of the form $4x^2+170x+1847$; as pointed out by the referee, this is reducible into Euler's form by putting $y=2x+42$. (Under the enumeration above, the horizontal, vertical or diagonal straight lines correspond to quadratic forms, which have a leading term $4x^2$.) We have tried
other “Peano numberings” of the lattice points. For example, (Fig. 2) for lattice points in the positive quadrant:

\[
(0, 0) \rightarrow 1, \ (0, 1) \rightarrow 2, \ (1, 1) \rightarrow 3, \ (1, 0) \rightarrow 4
\]

\[
(0, 2) \rightarrow 5, \ (1, 2) \rightarrow 6; \ (2, 2) \rightarrow 7, \ (2, 1) \rightarrow 8, \ (2, 0) \rightarrow 9, \text{ etc.}
\]
The successive points on the principal lines will ultimately have coordinates given by quadratic forms with leading term $1 \cdot x^2$. Or, (Fig. 3) for points in the upper half plane:

$$(0, 0) \to 1, \ (-1, 0) \to 2, \ (-1, 1) \to 3, \ (0, 1) \to 4$$

$$(1, 1) \to 5, \ (1, 0) \to 6, \ (-2, 0) \to 7, \ (-2, 1) \to 8$$

$$(2, 2) \to 9, \ (-1, 2) \to 10, \ (0, 2) \to 11, \ (1, 2) \to 12$$

$$(2, 2) \to 13, \ (2, 1) \to 14, \ (2, 0) \to 15$$

There, the principal straight lines correspond to forms with the term $2x^2$.

The obvious questions, e.g.: Is the distribution of points of $P$ asymptotically symmetric in every angle from the origin? Are there lines containing infinitely many primes? What is the asymptotic density of points on lines (e.g. are there pairs of nonfactorable lines with different asymptotic densities)? etc. seem to be hardly answerable with the present knowledge of the distribution of primes.

We have observed many nonfactorable but, so to say, “almost factorable” lines, i.e. lines extremely poor in primes. We should add, as a curiosity, that as we displayed similarly the set $L$ of lucky numbers (see [1]), in (Fig. 4) (the numbering of lattice points in this case is by a “discontinuous spiral,” i.e. as in (Fig. 3) but going through both half planes); it appears again that there is a great deal of “structure”; in particular, some of the principal lines are manifest. This is much more surprising, of course, since there is no obvious multiplicative property of the set of luckies and no relation which is rigorous between the divisibility of quadratic forms and the definition of the sieve determining the lucky numbers.
The first observation of the properties of the \( P \) set was made on a few hundred points by hand. On the electronic computing machine "Maniac II" in Los Alamos we have been able to use a scope attached to the machine, which can display up to 65,000 points obtained as a result of calculation. This is then photographed and our pictures show a few of the results. We have magnetic tapes containing tables of primes up to ninety million. After discovering the quadratic forms which seem to be rich in primes up to \( n = 100,000 \) or so, we then investigated primes up to ten million for such forms (see [2]). A few of the statistics are given below:

For primes in the Euler form \( n = x^2 + x + 41 \) we found the ratio \( r \) of these to all numbers of this form \( n \) up to 10,000,000 to be \( r = .475 \ldots \).

1. For the form \( n = 4x^2 + 170x + 1847 \) there are 727 primes in the first 1560 for numbers of this form \((1 \leq n \leq 10,000,000) (r = .466)\).

2. For \( n = 4x^2 + 4x + 59 \) yields \( r = .437 \ldots \).

3. For \( n = 2x^2 + 4x + 117 \) (the "rare" form) \( r = .050 \ldots \).

(A reason for rarity is that for no prime \( p < 29 \) is divisibility by \( p \) excluded)

4. The quadratic form \( n = x^2 + x + 1 \) is rich in "luckies." Up to numbers \( n \leq 300,000 \) \( r = .29 \ldots \).

This work was performed under the auspices of the U. S. Atomic Energy Commission.

References


ON THE MEAN VALUES OF INTEGRAL FUNCTIONS AND THEIR DERIVATIVES DEFINED BY DIRICHLET SERIES

J. S. GUPTA, Indian Institute of Technology, Kanpur, India

1. Consider the Dirichlet series \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \), where \( \lambda_{n+1} > \lambda_n \), \( \lambda_1 \geq 0 \), \( \lim_{n \to \infty} \lambda_n = \infty \), \( s = \sigma + it \) and

\[
(1.1) \quad \lim_{n \to \infty} \frac{\log n}{\lambda_n} = 0.
\]

Let \( \sigma_e \) and \( \sigma_a \) be the abscissa of convergence and the abscissa of absolute convergence, respectively, of \( f(s) \). Let \( \sigma_e = \infty \) then \( \sigma_a \) will also be infinite, since according to a known result ([1], p. 4) a Dirichlet series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore \( f(s) \) represents an integral function.

We define the mean values of \( f(s) \) as

\[
I_r(\sigma) = I_r(\sigma, f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^r \, dt, \quad (r > 0),
\]