Chapter 8. Model of the Accelerometer

8.1 The static model
8.2 The dynamic model
8.3 Sensor System simulation
8.3 Sensor System Simulation

In order to predict the behavior of the mechanical sensor in combination with the electronic circuits, an electrical analogon is derived for the readout of the mass position and feedback of electrostatic forces.

Two distinct modes of accelerometer will be investigated separately:

8.3.1 Normal motion of the seismic mass
8.3.2 Tilt motion of the seismic mass
8.3.1 Normal motion of the seismic mass

Electrical Analogy

<table>
<thead>
<tr>
<th>Mechanical Quantity</th>
<th>Electrical Quantity</th>
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</thead>
<tbody>
<tr>
<td>v [m s⁻¹]</td>
<td>U [V]</td>
</tr>
<tr>
<td>F [N]</td>
<td>I [A]</td>
</tr>
</tbody>
</table>

where mass M Ü capacitor C with unit [kg]
Spring 1/K z Ü inductor L with unit [mN⁻¹]
pure damper 1/f Ü resistor R with unit [ms⁻¹N⁻¹]

- The position of the mass can be found by integrating the velocity, which is equivalent to the flux \( \phi \) \( (\phi = \int U \, dt) \) in the electrical domain. The equivalent differential equation becomes:

\[
C \frac{dU}{dt} + \frac{1}{R} U + \frac{1}{L} \int U dt = I = -Ma_z
\]

\[
\Rightarrow \quad C \frac{d^2 \phi}{dt^2} + \frac{1}{R} \frac{d\phi}{dt} + \frac{1}{L} \phi = I = -Ma_z
\]
Electrical-analogy circuit of normal motion

**Fig. 2-29** Basic electrical-analogon describing the vertical movement of the seismic mass due to vertical accelerations, disregarding the compressibility effect of the gas film.

RLC circuit:

\[
C\frac{dU}{dt} + \frac{1}{R} U + \frac{1}{L} \int U dt = I = -Ma_z
\]
(1) In electrical circuit theory, it is well known that the contribution of R to the noise force (current) is:

\[ \bar{i}_n^2 = \frac{4k_B T}{R_1} = 4k_B T f_z \]

where \( k_B \) is Boltzman constant (=1.38x10\(^{23} \) J/K), \( T \) is absolute temperature and \( R \) is the resistor value. \( f_z \) is the coefficient relating the damping force in the z-direction.

(2) At higher frequencies, the compressibility effect of the gas-film, resulting in a higher spring force and a lower damping force, can be modeled by expanding the resistor, which represents the incompressible solution of the squeeze-damping mechanism, into several shunt connections of resistors and inductors.

The resistors represent the damping behavior, whereas the inductors indicate the compressible spring-like behavior of the gas film.

![Basic electrical-analogon describing the vertical movement of the seismic mass due to vertical accelerations, disregarding the compressibility effect of the gas film.](image)
Normal Motion

\[ F_0 = \frac{64 P_a b \sigma h}{\pi^6 h_0} \sum_{m,n \text{ odd}} \frac{m^2 + (n/\beta)^2}{(mn)^2 \left[ (m^2 + (n/\beta)^2)^2 + \sigma^2/\pi^4 \right]} \]

\[ F_1 = \frac{64 P_a b \sigma^2 h}{\pi^3 h_0} \sum_{m,n \text{ odd}} \frac{1}{(mn)^2 \left[ (m^2 + (n/\beta)^2)^2 + \sigma^2/\pi^4 \right]} \]

\[ M_{0, \theta} = M_{1, \theta} = M_{1, \theta_x} = M_{1, \theta_y} = 0 \] (2-65)

Damping force

\[ F_0 = f v \]

Spring force

\[ F_1 = k_z Z \]

\[ \sigma = \frac{12 \mu a^2 \omega}{P_a h_0^2} \]

\[ R = 1/f = v/F_0 \]

\[ L = 1/k_z = \delta h/F_1 \]

\[ L_{top,m,n} = \frac{\pi^4 h_0}{64 a_{top} b_{top} P_a} (mn)^2 \quad ; \quad m, n \text{ odd} \]

\[ R_{top,m,n} = \frac{\pi^6 h_0^3}{768 a_{top}^3 b_{top} P_a} (mn)^2 \left[ m^2 + (n/\beta_{top})^2 \right] \quad ; \quad m, n \text{ odd} \]

\[ L_{bot,m,n} = \frac{\pi^4 h_0}{64 a_{bot} b_{bot} P_a} (mn)^2 \quad ; \quad m, n \text{ odd} \]

\[ R_{bot,m,n} = \frac{\pi^6 h_0^3}{768 a_{bot}^3 b_{bot} P_a} (mn)^2 \left[ m^2 + (n/\beta_{bot})^2 \right] \quad ; \quad m, n \text{ odd} \]

where \( a_{top}, a_{bot}, b_{top}, b_{bot} \) are the dimensions of the seismic mass at the top and bottom and \( \beta_{top} = b_{top}/a_{top} \) and \( \beta_{bot} = b_{bot}/a_{bot} \) are the aspect ratios of the top and bottom of the seismic mass. In case of the accelerometer, \( a_{top} = b_{top} = c_1 \) and \( a_{bot} = b_{bot} = c_1 - t_{wafer, \sqrt{2}} \) (ref. Figure 2-2), so that \( \beta_{top} = \beta_{bot} = 1 \).
(3) To show the influence of the compressibility effect on the dynamic behavior of the accelerometer.

- The smaller the gap, the more the device is damped and the lower the bandwidth of the device will be.
- The cut-off frequency of the squeeze film damping mechanism is reduced with reduce gap.
- The peak in the frequency response which occurs beyond the bandwidth of the accelerometer increases with the decrease of the gap.

![Simulated frequency response of the accelerometer, undergoing a normal acceleration, including the compressibility effect of the gas-film.](image)

\[
M = 1.616 \times 10^{-6} \text{ kg and } K_z = 315 \text{ Nm}^{-1}
\]

\[
P_a = 10^5 \text{ Pa, } \mu = 1.8 \times 10^{-5} \text{ Pa s}
\]

\[
a = 2 \text{ mm, } \delta h = 1 \mu \text{m}
\]
(4) By approximating the squeeze-film damping of the gap by one RL-selection for the top and bottom of the film.

Fig. 2-32 Electrical analogon model of the accelerometer, using a one-term approximation of the squeeze-film damping at the top and bottom of the mass.

The transfer function:

\[
\frac{\Delta z}{a_z} = -\frac{M}{Ms^2 + K_z + \frac{s}{sL_{top,1,1} + R_{top,1,1}} + \frac{s}{sL_{bot,1,1} + R_{bot,1,1}}}
\]

4th order system:

4 poles: \( p_1, p_2, p_3, p_4 \)

2 zeros: \( z_1, z_2 \)

\[
z_1 = -\frac{R_{top,1,1}}{L_{top,1,1}} = -\frac{\pi^2 P_a h_0^2}{12\mu} \left( \frac{1}{a_{top}^2} + \frac{1}{b_{top}^2} \right)
\]

\[
z_2 = -\frac{R_{bot,1,1}}{L_{bot,1,1}} = -\frac{\pi^2 P_a h_0^2}{12\mu} \left( \frac{1}{a_{bot}^2} + \frac{1}{b_{bot}^2} \right)
\]
8.3.2 Tilt motion of the seismic mass

In case of an arbitrary acceleration \((a_x, a_y, a_z)\), the model has to include the rotation of the mass around the x and y-axis.

### Electrical Analogy

<table>
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<tr>
<td>(\omega) [rad s(^{-1})]</td>
<td>(U) [V]</td>
</tr>
<tr>
<td>(\tau) [Nm]</td>
<td>(I) [A]</td>
</tr>
</tbody>
</table>

- The angle of rotation can be found by integrating the angular velocity, which is equivalent to the flux \(\phi (\phi = \int U \, dt)\) in the electrical domain. The equivalent differential equation becomes:

\[
C \frac{dU}{dt} + \frac{1}{R} U + \frac{1}{L} \int U \, dt = I = -Ma_xZ_c \Rightarrow C \frac{d^2\phi}{dt^2} + \frac{1}{R} \frac{d\phi}{dt} + \frac{1}{L} \phi = I = -Ma_xZ_c
\]
Electrical-analogy circuit of tilt motion

(1) If the mass rotations are small. There is no coupling between the several modes of the seismic mass. In that case, the described analogy model is valid also for a combination of normal and lateral accelerations.

Fig. 2-37 Basic electrical analogous circuit describing the tilt motion of the seismic mass due to lateral accelerations, disregarding the compressibility of the gas-film.

RLC circuit:

\[ C_2 \frac{dU}{dt} + \frac{1}{R_2} U + \frac{1}{L_2} \int U dt = I = -Ma_x Z_c \]

\[ C_3 \frac{dU}{dt} + \frac{1}{R_3} U + \frac{1}{L_3} \int U dt = I = -Ma_y Z_c \]
The gas-film damping effect by the tilt motion can be modeled by using a shunt connection of several resistors and inductors.

\[
M_{0, \theta_x} = \frac{16P_a b^3 \sigma \theta_x}{\pi^6} \sum_{m \text{ even}} \frac{m^2 + (n/\beta)^2}{h_0 (mn)^2 \left\{ [m^2 + (n/\beta)^2] + \sigma^2/\pi^4 \right\}}
\]

\[
M_{1, \theta_y} = \frac{16P_a b^3 \sigma^2 \theta_y}{\pi^8} \sum_{m \text{ odd}} \frac{1}{h_0 (mn)^2 \left\{ [m^2 + (n/\beta)^2] + \sigma^2/\pi^4 \right\}}
\]

\[
M_{0, \theta_y} = \frac{16P_a a^3 b \sigma \theta_x}{\pi^6} \sum_{m \text{ odd}} \frac{m^2 + (n/\beta)^2}{h_0 (mn)^2 \left\{ [m^2 + (n/\beta)^2] + \sigma^2/\pi^4 \right\}}
\]

\[
M_{1, \theta_y} = \frac{16P_a a^3 b \sigma^2 \theta_y}{\pi^8} \sum_{m \text{ even}} \frac{1}{h_0 (mn)^2 \left\{ [m^2 + (n/\beta)^2] + \sigma^2/\pi^4 \right\}}
\]

\[
F_0 = F_1 = 0 \quad (2-73)
\]

\[
L_{\theta_x, \text{top.m.n}} = \frac{\pi^4 h_0}{16a_{\text{top}} b_{\text{top}}^3 P_a} (mn)^2 \quad ; \text{m odd, n even}
\]

\[
R_{\theta_x, \text{top.m.n}} = \frac{\pi^6 h_0^3}{192a_{\text{top}}^3 b_{\text{top}}^3 \mu} (mn)^2 [m^2 + (n/\beta_{\text{top}})^2] \quad ; \text{m odd, n even}
\]

\[
L_{\theta_y, \text{bot.m.n}} = \frac{\pi^4 h_0}{16a_{\text{bot}} b_{\text{bot}}^3 P_a} (mn)^2 \quad ; \text{m odd, n even}
\]

\[
R_{\theta_y, \text{bot.m.n}} = \frac{\pi^6 h_0^3}{192a_{\text{bot}}^3 b_{\text{bot}}^3 \mu} (mn)^2 [m^2 + (n/\beta_{\text{bot}})^2] \quad ; \text{m odd, n even}
\]

where \(a_{\text{top}}, a_{\text{bot}}, b_{\text{top}}, b_{\text{bot}}\) are the dimensions of the seismic mass at the top and bottom and \(\beta_{\text{top}} = b_{\text{top}}/a_{\text{top}}\) and \(\beta_{\text{bot}} = b_{\text{bot}}/a_{\text{bot}}\) are the aspect ratios of the top and bottom of the seismic mass. In case of the accelerometer, \(a_{\text{top}} = b_{\text{top}} = c_1\) and \(a_{\text{bot}} = b_{\text{bot}} = c_1 - t_{\text{wafer}}\sqrt{2}\) (ref. Figure 2-2), so that \(\beta_{\text{top}} = \beta_{\text{bot}} = 1\).
• The bandwidth of the lateral movement is larger than the bandwidth of the normal movement of the seismic mass.
• The smaller the gap, the more the device is damped and the lower the bandwidth of the device will be.
• The peak in the frequency response occurs at small values of the gap.

$v_{Si} = 0.0625, \ t_{wafer} = 525 \ \mu m, \ l_b = 380 \ \mu m, \ w_b = 200 \ \mu m,$
$h_b = 4 \ \mu m, \ c_1 = 4 \ mm, \ a_1 = 2 \ mm, \ a_2 = 1.7 \ mm. \ E=1.69\cdot10^{11}Nm^{-2}.$

The thickness of the used beams is determined by the thickness of the epi-layer, which in our case is 4 \ \mu m. When using a beam width of 200 \ \mu m and a beam length of 380 \ \mu m, the sensitivities are:

• $K_z = 315 \ \text{Nm}^{-1}$
• $K_{\theta_x} = K_{\theta_y} = 1.27 \cdot 10^{-3} \ \text{Nm}$
In the case of 20µm, the translational movement of the mass can be over-damped, whereas the tilt movement of the mass is under-damped. Consequently, the undesired tilt movement can affect the dynamic performance of the device.

When using servo-accelerometer, the gap between the mass and the encapsulation will be small, so that the translation and tilt movements of the mass will be highly over-damped.
Conclusion

• The static mass movement of the seismic mass at a certain applied acceleration is determined by the weight of the mass and the beam suspension.

• In order to retain a linear suspension, the displacements of the mass must be kept small with respect to the thickness of the beams.

• The residual stress in the structure should be avoided.

• The static equations of the accelerometer show the uncoupled modes between the movement in the z-direction and the rotations of the mass around the x and y axis.

• The optimal geometry is found by minimizing the cross axis sensitivity which requires the beams to be placed near the corners of the seismic mass, oriented in all <110> direction.

• Both vertical and the tilt motion of the seismic mass can be damped using squeeze film damping mechanism.

• The damping can be controlled by the mass size, gap size, gas mixture and pressure.

• Due to the compressibility of the gas, the bandwidth of the damping mechanism is limited.

• The electronic analogon model of the sensor is able to simulate the interaction of the mechanical sensor and electronic circuit to be connected to it.