GENERAL INCENTIVES IN FINITE GAME THEORY

BY

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DISSERTATION

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Abstract

A general framework for analyzing finite games will be introduced. The concept of an incentive function will be defined so that it is compatible with the updating protocol defined in Nash’s proof of existence of equilibrium in all finite games. A general notion of incentive equilibrium will be defined as the fixed points of the updating protocol. It will be shown that given a continuous incentive, an incentive equilibrium will exist for any finite game. Specific examples will be given that have connections to canonical game dynamics and the incentive equilibrium are described fully. Non canonical examples will also be defined including an example based on simultaneous updating of strategies by the agents.

A system of differential equations will be derived from the updating protocol, which have fixed points exactly at incentive equilibrium. It will be shown that the canonical dynamics can be achieved using the canonical incentives. Specifically the Brown-von Neumann-Nash dynamics, replicator equations, projection dynamics, logit equations, best-reply dynamics, and pairwise comparison dynamics will all be derived from their respective incentives. It will be shown that the incentive dynamics are fully general in the sense that it can be used to describe all possible game dynamics that preserve the strategy space.

Incentive stable states, ISS, for general incentive dynamics will be defined as an analog to the concept of the evolutionary stable states, ESS, in the replicator dynamics. The connection between the replicator dynamics and information theory is discussed. Of particular interest is the use of the Kullback-Leibler divergence as a Lyapunov function to show that an ESS is asymptotically stable for the replicator dynamics. It will be shown that the Kullback-Liebler divergence is also a Lyapunov function for the incentive dynamics at ISS.
An important example of ISS is given by realizing the uniform distribution as an interior ISS for an incentive based on simultaneous updating in a class of games that includes all variations of Rock-Paper-Scissors, RPS. The uniform distribution is the unique incentive (Nash) equilibrium for all of the canonical dynamics. However, for a specific choice of parameters in RPS if an orbit has an initial condition that is not the uniform distribution it will not converge to the unique fixed point in any of the canonical dynamics. In stark contrast, the ISS condition guarantees the uniform distribution is asymptotically stable for simultaneous updating. It will be shown that it is in fact globally asymptotically stable for the interior of the strategy space.

A collection of numerical results will be given for a particular incentive. It will be shown in a number of distinct games that the incentive equilibrium is a better approximation to human behavior than the Nash equilibrium. This collection of games includes the Prisoner’s Dilemma, Matching Pennies, Battle of the Sexes, the Traveler’s Dilemma, Chicken, and other unnamed games of interest. Several different variations on each one of these games will be given to demonstrate the dependence on absolute differences in payoffs inherent in this model of incentive. This feature, known as cardinal dependence, is exhibited by human actors but is missing from the Nash model, which is preference, or ordinal, dependent. It will also be shown that under this incentive agents display a competitive nature as evidenced by certain games with ‘win-win’ strategies having alternative equilibrium that are asymptotically stable. Specifically this behavior seems to appear when one agent has more opportunities to attain its maximum payoff than the other agents. This is also an observable behavior of human actors which is absent from the Nash model.
For Cleo Arthur
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Chapter 1

Introduction

I can calculate the motions of erratic bodies, but not the madness of a multitude. [Fra] - Sir Issac Newton

These are famous words of lament from one of our greatest geniuses after he lost a fortune investing in the South Sea Bubble of 1720. This and innumerable other phenomenon in economics and social behavior provide us with the impetus for developing a theory of social decision making. After all, if the father of classical mechanics can fall prey to what in modern terms we might refer to as ‘irrational exuberance’ [Gre96], what hope is there without some precise notion of how we as humans act or perhaps more importantly how we do act? We will now discuss some of the early results of game theory and conclude with an outline for a general framework for games and behavior.

1.1 The Minimax Theorem

As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved - John von Neumann [VNF53]

von Neumann and Morgenstern created the foundation for the modern concept of equilibrium in games in the classic book “The Theory of Games and Economic Behavior” [vNMRK07]. They sought to find a solution concept for zero sum games, as these form an important class of games in economics. The minimax strategy of von Neumann [vN28]
satisfied all of their axioms of rational behavior in agents. These strategies have the characteristic that the utility or payoff to the agent is as large as possible while assuming the least amount of risk. For zero sum games this reduces to minimizing one’s own maximum payoff.

Unfortunately, the minimax concept is only defined for two person games. To account for this, von Neumann assumed that agents would form coalitions against each other and in this sense the problem would reduce to a two coalition game to which the minimax solution applied.

1.2 The Nash Equilibrium

Nash sought to generalize the notion of a minimax solution and to show that a solution exists for any finite game, zero-sum or not, without the need for cooperation. In his seminal papers [Nas50, Nas51] he defined, quite elegantly, a solution concept for all $n$-person finite games that would eventually earn him the Nobel Memorial Prize in Economic Sciences. The concept generalized the work of von Neumann by defining an equilibrium point to be a collection of strategies where every agent is attaining its maximum payoff given that all the other agents hold their strategy fixed. He then very cleverly applied fixed point theorems of Kakutani [Kak41] and Brouwer respectively.

Since its inception, the Nash equilibrium has been used in many fields including but not limited to; economics, computer science, political science, psychology, and biology. In many of these fields the solution seems to be a reasonable approximation to the observable behavior. This is especially true in instances where humans are not the actors in the game. This is disconcerting as game theory was primarily developed to understand problems in economics, where human agents abound. There are decades worth of data describing this unfortunate phenomenon, see [Bas07, CCG07, Bas94, RC65, GR72] for examples, leaving game theorists, economists and the like, to wonder what humans actually use to make decisions.
1.3 Evolutionary Game Theory

Evolutionary game theory as developed in [TJ78, HS88, Wei97, HS98, Cre03, Now06, San10] is the study of evolutionary processes on a population. These evolutionary processes, natural selection for instance, are modeled by systems of differential equations, where an orbit of the equation can be thought of as the evolution of a population. These are games in that the fitness of a type within a population is dependent on the frequency of other types in the population. The types are directly analogous to the agents in classic game theory, while the frequencies of these types are essentially the strategies, and fitness is utility. The main difference between classic and evolutionary games is the population effectively plays the game against itself. Because of this, games are generally symmetric.

Perhaps the most important concept in evolutionary game theory is that of the evolutionary stable strategies (ESS) which was introduced by Maynard Smith and G R Price [S\textsuperscript{+}74, MSP73]. A population is at an ESS if it cannot be invaded by a sufficiently small mutant population. It has been shown [HS98, Har11] in a number of ways that ESS are asymptotically stable for the replicator equations [TJ78], which are by far the most widely recognized equations in evolutionary game theory. While these concepts have been widely adapted to problems with human actors, they still rely heavily on the Nash equilibrium\textsuperscript{1} and are thus prone to the same inaccuracies. To make matters slightly worse it can also be shown that there are instances when the dynamics do not converge to unique equilibrium points.

1.4 Outline of Mathematical Framework

We will begin the process of defining a general framework for analyzing finite games with the concept of an incentive function. Essentially incentive functions will measure the amount an agent would prefer to play a pure strategy versus the current strategy. We will define them such that it is compatible with the updating protocol defined in Nash’s proof of existence

\textsuperscript{1}All ESS are Nash equilibrium.
of equilibrium in all finite games [Nas51]. From these incentive functions and the updating protocol we will define a general concept of incentive equilibrium. These will effectively be defined as the fixed points of the updating protocol and are generalizations of the Nash equilibrium. It will be shown that given a continuous incentive an incentive equilibrium will exist for any finite game. Specific examples will be given that have connections to canonical game dynamics and their incentive equilibria will be described fully. Non canonical examples will also be defined including an example based on simultaneous updating of strategies by the agents. This incentive function will be analyzed throughout this dissertation as it seems to give solutions that are reasonable approximations to human behavior.

From the updating protocol, a system of differential equations will be derived, which have fixed points exactly at incentive equilibrium. It will be shown that from the canonical incentives, the canonical dynamics can be achieved. Specifically the Brown-von Neumann-Nash dynamics [BvN50], replicator equations [TJ78], projection dynamics [SDL08], logit equations [FL98], best-reply dynamics [GM91], and pairwise comparison dynamics [Smi84] will all be derived from their respective incentives. It will be shown that the incentive dynamics are fully general in the sense that it can be used to describe all possible game dynamics that preserve the strategy space.

Incentive stable states, ISS, for general incentive dynamics will be defined as an analog to the concept of the evolutionary stable states, ESS, in the replicator dynamics. The connection between the replicator dynamics and information theory is discussed. Of particular interest is the use of the Kullback-Leibler divergence [KL51] as a Lyapunov function to show that an ESS is asymptotically stable for the replicator dynamics. It will be shown that the Kullback-Liebler divergence is also a Lyapunov function for the incentive dynamics at ISS.

An important example of ISS is given by realizing the uniform distribution as an interior ISS for an incentive based on simultaneous updating in a class of games that includes all variations of Rock-Paper-Scissors, RPS. The uniform distribution is the unique incentive (Nash) equilibrium for all of the canonical dynamics. However, for a specific choice of
parameters in RPS, all of the canonical dynamics exhibit the same behavior: if an orbit has an initial condition that is not the uniform distribution it will not converge to the unique fixed point. Instead they are have a stable limit cycle which is close to what is referred to as the Shapley\(^2\) triangle. In stark contrast, the ISS condition for our dynamic guarantees the uniform distribution is asymptotically stable. It will be shown that it is in fact globally asymptotically stable for the interior of the strategy space.

A collection of numerical results will be given for our simultaneous updating incentive. It will be shown in a number of distinct games that the incentive equilibria are a better approximation to human behavior than the Nash equilibria. This collection of games includes the Prisoner’s Dilemma, Matching Pennies, Battle of the Sexes, the Traveler’s Dilemma, Chicken, and other unnamed games of interest. Several different variations on each one of these games will be given to demonstrate the dependence on absolute differences in payoffs inherent in this model of incentive. This feature, known as cardinal dependence, is exhibited by human actors but is missing from the Nash model, which exhibits ordinal dependence. It will also be shown that under this incentive agents display a competitive nature as evidenced by certain games with ‘win-win’ strategies having alternative equilibrium that are asymptotically stable. Specifically this behavior seems to appear when one agent has more opportunities to attain its maximum payoff than the other agents. This is also an observable behavior of human actors which is absent from the Nash model.

\section*{1.4.1 Notation and Definitions}

We shall denote the finite set of agents by \(N = \{1, 2, \ldots, n\}\) for some \(n \in \mathbb{N}\). Each agent \(i\) is endowed with a finite set of pure strategies, which will be denoted \(S_i = \{1, 2, \ldots, s_i\}\), with \(s_i \in \mathbb{N}\) as well. To allow the agents to mix their strategies, they may choose strategies from

\(^2\)Named in honor of Lloyd Stowell Shapley
the simplex on $s_i$ vertices,

$$\Delta_i = \left\{ x_i \in \mathbb{R}^{s_i} \left| x_{i\alpha} \geq 0, \sum_{\alpha} x_{i\alpha} = 1 \right. \right\}. $$

which is the convex hull of $S_i$, or equivalently the space of probability distributions over the finite set. For simplicity we will embed $S_i$ in $\Delta_i$ such that $\alpha \in S_i \mapsto e_{i\alpha} \in \mathbb{R}^{s_i}$ where $e_{ik}$ is the $k$th standard unit vector in the Euclidean space $\mathbb{R}^{s_i}$. We denote $S = \times_i S_i$ and $\Delta = \times_i \Delta_i$ as the pure and mixed strategy spaces respectively for the game.

It is often convenient to denote the pure and mixed strategy spaces without a particular player; $S_{-i}$, and $\Delta_{-i}$ respectively. We define $S_{-i} = \times_{j \neq i} S_j$, and $\Delta_{-i} = \times_{j \neq i} \Delta_j$. Elements in these sets can be interpreted many different ways. In particular $S_{-i}$ is a $s_{-i} = \frac{|S|}{s_i}$ dimensional space and we would prefer to identify elements in this space with standard unit vectors in $\mathbb{R}^{s_{-i}}$ as before. Unfortunately, there are $s_{-i}$! ways to accomplish this. In practice, we will only use this identification when we will sum over all possible combinations of pure strategies. Using a different identification will simply result in a permutation of terms in a finite sum, which of course has no effect. $k \in S_{-i}$ is a multi-index given by $(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$. Our embedding, given by $k \in S_{-i} \mapsto e_{-i\beta} \in \mathbb{R}^{s_{-i}}$, extends to $\Delta_{-i}$ such that $x_{-i} \in \Delta_{-i} = \sum_{\beta} x_{-i\beta} e_{-i\beta}$ with $x_{-i\beta} = \prod_{j \neq i} x_{jk_j}$. If we have a single agent we will interpret $S_{-i}$, $\Delta_{-i}$, $s_{-i}$, and $x_{-i}$ as $S$, $\Delta$, $s$ and $x$ respectively.

We will also adopt a convention for replacement for part of a strategy profile, $x \in \Delta$. We write $(t_i, x_{-i}) \in \Delta = (x_1, x_2, \ldots, t_i, \ldots, x_n)$, where the $i$th component of $x$ has been replaced by another strategy $t_i \in \Delta_i$.

Each agent will have a utility function defined over the set of all possible combinations of pure strategies $S$. We will denote this utility

$$u_i : S \rightarrow \mathbb{R}.$$
These utility functions have unique $n$-linear extensions to $\Delta$ given by

$$u_i(x) = \sum_{\alpha} x_{i\alpha} u_i(e_{i\alpha}, x_{-i}) = x_i^T A_i x_{-i},$$

where $a_{i\alpha\beta} = u_i(e_{i\alpha}, e_{-i\beta})$ and $A_i = \{a_{i\alpha\beta}\}$. We will simply refer to these extensions as the utility functions from now on.
Chapter 2

A General Framework for Equilibrium in Finite Games

The Nash equilibrium [Nas50] is ubiquitous throughout game theory. The rise of evolutionary game theory has put new emphasis on dynamics of rationality as opposed to static equilibrium concepts. Most of these dynamic models are focused on either the Nash equilibrium itself or some refinement of the Nash equilibrium, e.g. evolutionary stable strategies (ESS) [S+74], $\epsilon$-equilibrium [Eve57], etc. However, the question of applicability of the Nash equilibrium to actual human actors is still open. Often, in practice, the Nash equilibrium does not approximate actual behavior in a given game; see Prisoner’s Dilemma [RC65] or Traveler’s Dilemma [Bas94, Bas07] for instance.

We open up the interpretation of an equilibrium by first generalizing the notion of incentive for an agent. In the sequel we will derive from this interpretation a family of differential equations that can account for different updating procedures used by agents. First however, we will show there exists equilibrium in games with general incentives requiring minimal conditions.

2.1 Incentive Equilibrium

We begin the treatment of general equilibrium by starting with Nash’s second proof of existence in finite games [Nas51].

**Definition 2.1.** A strategy profile $x \in \Delta$ is a *Nash equilibrium* if and only if

$$u_i(x) = \max_{t \in \Delta_i} u_i(t, x_{-i}), \quad \forall i.$$
We may simplify this definition by further linearly extending the utility functions to all of $\mathbb{R}^m$, where $m = \prod_i s_i$. This results in $m$-linear functions which are harmonic on all of $\mathbb{R}^m$. We may therefore invoke the maximum principle on the closed convex space $\Delta$ recursively to deduce that the $u_i$’s are maximized (and minimized) in $S$. Therefore,

$$\max_{t \in \Delta_i} u_i(t, x_{-i}) = \max_{\alpha} u_i(e_\alpha, x_{-i}),$$

and thus we can give an equivalent definition for the Nash equilibrium as follows:

**Definition 2.2.** A strategy profile $x \in \Delta$ is a *Nash equilibrium* if and only if

$$u_i(x) = \max_{\alpha} u_i(e_\alpha, x_{-i}), \forall i.$$

Thus it was natural for Nash to define a class of continuous incentive functions by

$$\varphi_{i\alpha}^{Nash}(x) = (u_i(e_\alpha, x_{-i}) - u_i(x))_+$$

where

$$(x)_+ = \max(0, x).$$

It is at this point where we are ready to define the updating protocol by which agents will discreetly change their strategies. We define the map

$$T(x) : \times_i \mathbb{R}^{s_i} \to \times_i \mathbb{R}^{s_i}$$

where

$$T(x)_i = \frac{x_i + \sum_{\alpha} \varphi_{i\alpha}(x)e_\alpha}{1 + \sum_{\beta} \varphi_{i\beta}(x)}.$$

It is easily verified that the sum of the coefficients of $T(x)_i$ is 1 if $x_i \in \Delta_i$, however, if $x_{i\alpha} = 0$ we must have $\varphi_{i\alpha}(x) \geq 0$ in order to preserve the simplex. We also require $\sum_{\beta} \varphi_{i\beta}(x) \neq -1$.
for any $x \in \times_i \mathbb{R}^{s_i}$. This leads us to our definition of generalized incentive.

**Definition 2.3.** A function $\varphi(x) : \times_i \mathbb{R}^{s_i} \rightarrow \times_i \mathbb{R}^{s_i}$ is an incentive function if and only if it satisfies both of the following conditions for all players $i$:

1. $x_{i\alpha} = 0 \Rightarrow \varphi_{i\alpha}(x) \geq 0$, $\forall \alpha$

2. $\sum_{\beta} \varphi_{i\beta}(x) \neq -1$ for any $x \in \times_i \mathbb{R}^{s_i}$.

If we have a function defined as above we may simply refer to it as the incentive for the game.

To complement our definition of incentive we must redefine equilibrium for the game to account for the general incentive. First, we will produce conditions for the mapping $T(x)$ to have a fixed point.

\[
0 = T(x)_i - x_i, \quad \forall i \tag{2.1}
\]

\[
= x_i + \frac{\sum_{\alpha} \varphi_{i\alpha}(x)e_{\alpha}}{1 + \sum_{\beta} \varphi_{i\beta}(x)} - x_i \tag{2.2}
\]

\[
= \frac{\sum_{\alpha} \varphi_{i\alpha}(x)e_{\alpha} - x_i \sum_{\beta} \varphi_{i\beta}(x)}{1 + \sum_{\beta} \varphi_{i\beta}(x)} \tag{2.3}
\]

\[
\Leftrightarrow \sum_{\alpha} \varphi_{i\alpha}(x)e_{\alpha} = x_i \sum_{\beta} \varphi_{i\beta}(x) \tag{2.4}
\]

\[
\Leftrightarrow \varphi_{i\alpha}(x) = x_{i\alpha} \sum_{\beta} \varphi_{i\beta}(x), \quad \forall i, \alpha \tag{2.5}
\]

Note that at a fixed point, 2.4 says that $\varphi_i$ is parallel to $x_i$. Furthermore, $\varphi_{i\alpha}(x)/x_{i\alpha}$ equals the total incentive provided that $x_{i\alpha} \neq 0$. If $\varphi_{i\alpha}(x) = 0$ at a fixed point then either $x_{i\alpha} = 0$ or $\sum_{\beta} \varphi_{i\beta}(x) = 0$, but $x_{i\alpha} = 0 \Rightarrow \varphi_{i\alpha}(x) = 0$. It is convenient then to abuse notation and write

\[
\frac{\varphi_i(x)}{x_i} = \left( \frac{\varphi_{i1}(x)}{x_{i1}}, \ldots, \frac{\varphi_{is_i}(x)}{x_{is_i}} \right)
\]

with the convention that $\varphi_{i\alpha}(x) = 0 \Rightarrow \frac{\varphi_{i\alpha}(x)}{x_{i\alpha}} = 0$. 

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**Definition 2.4.** A strategy profile \( \hat{x} \) is an *incentive equilibrium* if and only if

\[
\hat{x}_i \cdot \frac{\phi_i(\hat{x})}{\hat{x}_i} = \max_{x_i \in \Delta_i} x_i \cdot \frac{\phi_i(x)}{x_i}, \quad \forall i.
\]

Note that if we maximize the right hand side of the equation with respect to \( x_i \) under the simplex constraint we must have \( \phi_{i\alpha}(\hat{x})/\hat{x}_{i\alpha} \) all equal. The left hand side is clearly \( \sum_{\beta} \phi_{i\beta}(\hat{x}) \). Therefore, \( T(\hat{x}) = \hat{x} \iff \hat{x} \) is an incentive equilibrium. An intuitive description of the concept is that agents will achieve equilibrium if they either have no incentive or their incentives are in line with their current strategy for the game.

The following lemma will be very useful for proving not only our main theorem that an incentive equilibrium exists in every finite game, but will also allow us to identify equilibrium points in games that have certain symmetries.

**Lemma 2.1.** If the incentive is continuous, a fixed point exists for \( T \) in any closed convex \( U \subset \times_i \mathbb{R}^{s_i} \) that is left invariant by \( T \).

**Proof.** Given the assumptions, \( T \) maps from \( U \) to \( U \) continuously and thus Brouwer’s fixed point theorem guarantees the existence of a fixed point for \( T \) in \( U \). \( \square \)

We now have all the tools necessary to prove the main theorem.

**Theorem 2.2.** If the incentive is continuous, an incentive equilibrium point \( \hat{x} \) exists for any finite game.

**Proof.** We have defined the incentive functions such that the updating protocol \( T(x) \) defined above is a continuous map from \( \Delta \) to \( \Delta \) and thus by our lemma, there exits an \( \hat{x} \in \Delta \) such that \( T(\hat{x}) = \hat{x} \). \( T(\hat{x}) = \hat{x} \iff \hat{x} \) is an incentive equilibrium. \( \square \)

Other consequences of our lemma can also be obtained quite simply. For example, suppose a two player game has the property that the incentive is continuous and \( \phi_1(x) = \phi_2(x) \) for every \( x \in \Delta \) such that \( x_1 = x_2 \). The closed convex subset \( U = \{ x \in \Delta | x_1 = x_2 \} \) is left
invariant by $T$ and thus an incentive equilibrium point exists in $U$. We can generalize this to symmetric $n$-player games. Denote the symmetric group on a finite set $X$ as $\text{Sym}(X)$.

**Proposition 2.3.** Suppose all players have the same pure strategy space, $S_1$. Let $U = \{ x \in \Delta | x_{i\alpha} = x_{i\sigma_i(\alpha)} \text{ for some } \sigma_i \in \text{Sym}(S_1) \}$. If $\varphi(x)$ is continuous and $\varphi_{1\alpha}(x) = \varphi_{i\sigma_i(\alpha)}(x)$ for every $x \in U$, then an incentive equilibrium exists in $U$.

**Proof.** $U$ is closed and convex and left invariant by $T$, thus our lemma guarantees the existence of a fixed point of $T$ in $U$ which is a subset of $\Delta$. Thus the fixed point is an incentive equilibrium. \qed

### 2.2 Examples

We will now discuss some specific examples of incentives.

#### 2.2.1 Canonical Examples

We will refer to a collection of incentives that have been well studied in other venues. They are all very closely related to the Nash equilibrium. In fact they all share the property that an interior Nash equilibrium is an incentive equilibrium.

**Nash Incentive**

Above it was noted that Nash defined a family of functions as

$$
\varphi_{i\alpha}^{\text{Nash}}(x) = (u_i(e_\alpha, x_{-i}) - u_i(x))_+
$$

for every player $i$ and every strategy $\alpha \in S_1$. $\varphi^{\text{Nash}}(x)$ is trivially an incentive function as it is non-negative in every component at every $x \in \Delta$. Clearly this incentive is continuous as $f(x) = (x)_+$ and $u_i(x)$ are both continuous. Thus an incentive equilibrium exists for every finite game.
We expect that the incentive equilibrium must in fact be a Nash equilibrium. If \( \hat{x} \) is a Nash equilibrium, \( \varphi(\hat{x}) = 0 \) and thus every Nash equilibrium is an incentive equilibrium. Conversely, if \( \hat{x} \) is an incentive equilibrium we have several possibilities. If \( \sum_\beta \varphi^{Nash}_i(\hat{x}) = 0 \), \( \hat{x} \) is a Nash equilibrium. It suffices then to consider the case when the sum is positive. Also we need not consider the case when \( \hat{x}_{i\alpha} = 0 \) as this occurs if and only if \( \varphi^{Nash}_{i\alpha}(\hat{x}) = 0 \). Thus we can assume \( \hat{x}_{i\alpha} > 0 \). This can occur in an incentive equilibrium if and only if \( \varphi^{Nash}_{i\alpha}(\hat{x}) > 0 \), which implies \( u_i(e_\alpha, \hat{x}_i) > u_i(\hat{x}) \) for any \( \alpha \) such that \( \hat{x}_{i\alpha} > 0 \). For these \( \alpha \) the inequality, \( \hat{x}_{i\alpha} u_i(e_\alpha, \hat{x}_i) > \hat{x}_{i\alpha} u_i(\hat{x}) \), must also hold. If we sum over these \( \alpha \) we obtain the impossible condition, \( u_i(\hat{x}) > u_i(\hat{x}) \). Thus at equilibrium \( \varphi^{Nash}(\hat{x}) = 0 \), which implies \( \hat{x} \) is a Nash equilibrium.

**Replicator Incentive**

Interestingly, the replicator dynamics [TJ78], specifically the \( n \)-population models, given by \( \dot{x}_{i\alpha} = x_{i\alpha} (u_i(e_\alpha, x_{-i}) - u_i(x)) \), provide incentive functions as well. Define \( \varphi^R_{i\alpha}(x) = x_{i\alpha} (u_i(e_\alpha, x_{-i}) - u_i(x)) \). \( \varphi^R(x) \) is an incentive function since \( \sum_\beta \varphi^R_{i\beta}(x) = 0 \) and \( x_{i\alpha} = 0 \Rightarrow \varphi^R_{i\beta}(x) = 0 \). The replicator incentive function is not just continuous but analytic and thus easily satisfies the condition for existence of incentive equilibrium.

The classification of these equilibrium points are quite easy given the total incentive is identically zero. We must have all \( \varphi^R_{i\alpha}(\hat{x}) = 0 \) if \( \hat{x} \) is an incentive equilibrium. These functions are zero in three cases; \( \hat{x}_{i\alpha} = 0 \), \( \hat{x}_{i\alpha} = 1 \), and \( u_i(e_\alpha, \hat{x}_{-i}) = u_i(\hat{x}) \). Thus in the interior of \( \Delta \) our equilibrium is a Nash equilibrium. In fact given the last condition all Nash equilibria are replicator incentive equilibria. Finally, the first two conditions tell us that all \( x \in S \) are equilibria in contrast to the Nash incentive.

We can actually use many different incentive functions to get the same behavior. The simplest of these is \( \varphi^R_{i\alpha}(x) = x_{i\alpha} u_i(e_\alpha, x_{-i}) \). Notice that the total incentive \( \sum_\beta \varphi^R_{i\beta}(x) = u_i(x) \) for every \( x \in \Delta \), which could violate the second condition for an incentive function if the utility for any player is ever \(-1\). However, we can translate our payoffs by arbitrary functions.
$g_i(x)$ in every component for each player $i$. Thus $\varphi_{ia}^R(x) = x_{ia}(u_i(e_\alpha, x_{-i}) + g_i(x))$ and $
olimits \sum_\alpha \varphi_{ia}^R(x) = u_i(x) + g_i(x)$. Furthermore, our equilibrium condition remains unchanged as $x_{ia}(u_i(e_\alpha, x_{-i}) + g_i(x)) = x_{ia}(u_i(x) + g_i(x))$ is satisfied if $x_{ia}$ is 0 or 1, or if $u_i(e_\alpha, x_{-i}) = u_i(x)$. Thus in any finite game we can translate payoffs without changing equilibrium and every finite game has a translated version where our function is a valid incentive. In only slight contrast to the previous case the equilibria occurs when $\varphi_{ia}^R(\hat{x}) \neq 0$. In general we can use $\varphi_{ia}^R(x) = x_{ia}(u_i(e_\alpha, x_{-i}) + g_i(x))$ as our incentive function as long as $g(x)$ translates each of the minimum payoffs to any value greater than $-1$ (or maximum payoffs to values less than $-1$).

**Projection Incentive**

The projection dynamic, originally introduced by Nagurney and Zhang [NZ96] and presented in the style of Lahkar and Sandholm [LS08], is given by

$$\dot{x}_i = F_i(x),$$

where

$$F_i(x) = \begin{cases} (Ax)_i - \frac{1}{|S(Ax, x)|} \sum_{j \in S(Ax, x)} (Ax)_j & \text{if } i \in S(Ax, x) \\ 0 & \text{otherwise,} \end{cases}$$

for a single population. $S(Ax, x)$ is the set of all strategies in the support of $x$ as well as any collection of strategies that maximize the average presented in the first case above. $F_i(x)$ is clearly an incentive function for the game as $\sum_i F_i(x) = 0$ for every $x \in \Delta$. When $x_i = 0$, $i \notin \text{supp}(x)$ so $F_i(x) = 0$ or $(Ax)_i$ itself must be maximal and thus $F_i(x) = 0$.

It is shown in Sandholm, Dokumaci, and Lahkar [SDL08] that the replicator dynamics and the projection dynamics share many features. Most important in this discussion are the interior equilibria, which they showed to be Nash equilibria just as is the case for the
replicator incentive. However, in this case the discontinuity at the boundary means the main theorem does not apply.

**Best Reply**

The best reply incentive is quite easy to understand. It was originally introduced by Gilboa and Matsui [GM91]. Nash defined a function $B(x)$ in his original proof of existence, which is a set valued function that returns all of the pure strategy best replies to the strategy $x$. For our purposes we will use the function used by Young [You01], where

$$BR_{i\alpha}(x) = \begin{cases} 
1 & \text{if } e_{i\alpha} \in B(x) \\
0 & \text{otherwise.}
\end{cases}$$

To make this a function, there is a tiebreaker assumed so that only one pure strategy is taken to be the best reply. This function is a valid incentive since $BR_{i\alpha}(x) \geq 0$ and $\sum_\alpha BR_{i\alpha}(x) = 1$ for every $x \in \Delta$. The main theorem does not apply directly, however the incentive equilibria for this are exactly Nash equilibria. Thus the existence of a Nash equilibrium in all finite games guarantees the existence of an incentive equilibrium for the best reply incentive.

**Logit Incentive**

The logit incentive was originally introduced as a smoothed version of best reply by Fudenberg and Levine [FL98]. The incentive is defined as

$$\varphi_{i\alpha}^L(x) = \frac{\exp(\eta^{-1}u(e_{i\alpha}, x_{-i}))}{\sum_\beta \exp(\eta^{-1}u(e_{i\beta}, x_{-i}))}$$

and is obviously a valid incentive function since $\varphi_{i\alpha}^L(x) > 0$ for every $x$ and $\sum_\beta \varphi_{i\beta}^L(x) = 1$. The incentive is continuous and thus the main theorem applies. The incentive equilibria are exactly the fixed points of $\varphi^L(x)$. 

It should be noted that as $\eta \to 0$ the incentive converges to the best reply incentive. On the other end of the spectrum as $\eta \to \infty$ this incentive approaches the zero incentive (see below). The variable $\eta$ is thought of as ‘noise’, and is essentially a measure of error in logic, much like the $\epsilon$ in the $\epsilon$-equilibrium (see below).

**Smith Incentive**

The Smith incentive was developed by Micheal J. Smith [Smi84] to describe traffic flows. He suggests as a reasonable assumption that the rate which drivers swap from one route, $\beta$, to another route, $\alpha$, is given by the proportion of drivers on route $\beta$ times any additional cost of using $\beta$ over $\alpha$. Thus we can interpret this as an incentive to switch to $\alpha$ as

$$\varphi^S_{i\alpha}(x) = \sum_{\gamma} x_{i\gamma}(u_i(e_{i\gamma}, x_{-i}) - u_i(e_{i\alpha}, x_{-i}))_+$$

where Smith would drop the $i$ as there is only a single population of drivers.

The above function is always non-negative and thus is a valid incentive function. It is also obviously continuous thus the main theorem applies. Any Nash equilibrium, $x$, is an incentive equilibrium as $x_{i\alpha} > 0 \Leftrightarrow u_i(e_{i\alpha}, x_{-i}) = \max_\gamma u_i(e_{i\gamma}, x_{-i})$ and thus the terms in the above sum are all zero. The converse however is not generally true. The set of incentive equilibria in this case is called Wardrop equilibria [War52].

### 2.2.2 Other Examples

We can also describe a number of non-canonical examples which may be of interest.

**The Zero Incentive**

The trivial, or zero, incentive is given by $\varphi(x) = 0$ for all $x \in \Delta$. The function is clearly a valid incentive and also trivially satisfies the conditions for the existence of an incentive equilibrium. This of course is not surprising as the zero incentive fixes every point in $\Delta$, and
thus all points are incentive equilibria. If only one agent uses the zero incentive to update, it would appear to the opposition that the agent is choosing at random. In fact all elements of $\Delta_i$ are equally likely under this incentive.

**$\epsilon$-Nash Incentive**

$\epsilon$-Nash equilibria was first introduced by Everett [Eve57].

**Definition 2.5.** For a fixed $\epsilon > 0$, $x \in \Delta$ is an $\epsilon$-equilibrium if and only if

$$u_i(x) \geq u_i(t_i, x_{-i}) - \epsilon \quad \forall t_i \in \Delta_i, i.$$ 

We can make a similar simplification to what we did for the Nash equilibrium. Instead of checking every $x_i \in \Delta_i$ it suffices to check only those strategies in $S_i$. We can therefore use the incentive function

$$\phi^t_{i\alpha}(x) = (u_i(e_{i\alpha}, x_{-i}) - u_i(x) - \epsilon)_{+}.$$ 

This is clearly an incentive as it is always non-negative. It is also continuous which ensures the existence of incentive equilibrium. Of course, we already know that a Nash equilibrium exists in all finite games and a Nash equilibrium is an $\epsilon$-equilibrium for every $\epsilon > 0$.

There are simple examples of games that are repeated indefinitely which do not have Nash equilibria, but do still have $\epsilon$-equilibria for some $\epsilon > 0$. While this is beyond the scope of this discussion it is worth mentioning. Within our scope are the finitely repeated prisoner’s dilemmas. In these games it can be shown that the strategies tit-for-tat [AH81] and grim trigger are $\epsilon$-equilibria for some positive $\epsilon$ which depends on the payoffs of the one shot games.
Simultaneous Updating

While the notion of the $\epsilon$-equilibrium is very useful, it adds a degree of freedom to the problem of finding equilibria. $\epsilon$ would have to be fit to data in order to make the model viable and must be changed for each new game.

We draw inspiration for a new model from Brams’ “Theory of Moves” [Bra94]. In his book he describes a solution concept for $2 \times 2$ games that is based on a hypothetical negotiation. It is assumed that the agents begin this negotiation at a point in $S$, then each player does a hypothetical check on what would happen if they moved to their alternative pure strategy. They assume the other player will also switch and this alternating changing of strategy continues until a cycle is complete\textsuperscript{1}. Then the agent, using backward induction on the cycle, decides whether or not to make the first move. The solutions are the collection of possible outcomes given the 4 possible starting positions, giving this the feel of a discrete dynamical system.

We define an incentive function that takes into account the other players’ possible reactions to an agent’s move. We notice that if all agents are updating simultaneously then we can be anywhere in $\Delta$. Recall that all of the utility functions are maximized (and minimized) in $S$, so we will only make comparisons on the boundary. Our incentive is defined as

$$\varphi_{i\alpha}(x) = \sum_{\gamma}(a_{i\alpha\gamma} - u_i(x))_+$$

$$= \sum_{\gamma}(u_i(e_{i\alpha}, x_{-i}) - u_i(x) + a_{i\alpha\gamma} - u_i(e_{i\alpha}, x_{-i}))_+.$$  \hspace{1cm} \text{(2.6)}

The function is a valid incentive since it is always non-negative, and is continuous which means an incentive equilibrium exists for all finite games. Further analysis of this incentive, including numerical results, will appear in the sequels. The incentive equilibria for $\varphi^D(x)$ that lie in $S$ are very easily classified: they must be win-win situations in the sense that all

\footnote{This procedure always takes four moves, as the last move by the opposition returns the negotiation to its original state}
players are achieving their maximum payoffs in the game.
Chapter 3
Incentive Dynamics

The emergence of the replicator equations of Taylor and Jonker [TJ78] has created a renewed interest in dynamic models of rational decision making. There are several examples of these sorts of models including, but not limited to, the logit equation [FL98], best reply dynamics [GM91], the Brown-von Neumann-Nash (BNN) equations [BvN50], projection dynamics [NZ96, SDL08], Smith dynamics [Smi84], and others. Sandholm [San10] derives a family of differential equations, referred to as mean dynamics, given by

\[ \dot{x}_i = \sum_{j \in S} x_j \rho_{ji}(u(x), x) - x_i \sum_{j \in S} \rho_{ij}(u(x), x) \]

to describe the inflow and outflow of agents to and from a type \( i \) within a single population. The \( \rho_{ij} \) are supposed to represent the conditional switch rate of an agent switching from type \( i \) to type \( j \). If one were to specify this probability appropriately then one can recover all of the canonical dynamics listed above.

We seek a similarly flexible model but with incentive as the governing concept. We will proceed in such a way as to derive the BNN equations as introduced by Brown and von Neumann. As we have seen we can describe general equilibrium in games by way of incentive functions. We then allow agents to update their strategies via a revision protocol, \( T(x) \) given by

\[ T_i(x) = \frac{x_i + \sum_\alpha \varphi_{i\alpha}(x) e_{i\alpha}}{1 + \sum_\beta \varphi_{i\beta}(x)} . \]

If we repeat this mapping we can think of it as a discrete time dynamical system defined recursively by \( x^t = T(x^{t-1}) \), where the superscript here is to denote the time step and not
3.1 Incentive Dynamics

Instead of working with the discrete time system above, we prefer to work with a continuous time differential equation if possible. To facilitate this endeavor we will redefine every incentive function to have a simple time dependence. That is

$$\tilde{\varphi}(x) := t\varphi(x).$$

However, any change to the incentive function must also have an effect on the revision protocol, thus we write

$$x'_i = T_i(x, t) := \frac{x_i + \sum_\alpha \tilde{\varphi}_{i\alpha}(x)e_{i\alpha}}{1 + \sum_\beta \varphi_{i\beta}(x)}$$

$$= \frac{x_i + t\sum_\alpha \varphi_{i\alpha}(x)e_{i\alpha}}{1 + t\sum_\beta \varphi_{i\beta}(x)}.$$

Furthermore, it is now possible to define the time derivative of $x_i$.

$$\dot{x}_i := \lim_{t \to 0} \frac{x'_i - x_i}{t}$$

$$= \lim_{t \to 0} \frac{x_i + t\sum_\alpha \varphi_{i\alpha}(x)e_{i\alpha} - x_i - tx_i\sum_\beta \varphi_{i\beta}(x)}{t + t^2\sum_\beta \varphi_{i\beta}(x)}$$

$$= \lim_{t \to 0} \frac{\sum_\alpha \varphi_{i\alpha}(x)e_{i\alpha} - x_i\sum_\beta \varphi_{i\beta}(x)}{1 + t\sum_\beta \varphi_{i\beta}(x)}$$

$$= \sum_\alpha \varphi_{i\alpha}(x)e_{i\alpha} - x_i\sum_\beta \varphi_{i\beta}(x).$$

In individual coordinates we can write our family of differential equations as

$$\dot{x}_{i\alpha} = \varphi_{i\alpha} - x_{i\alpha}\sum_\beta \varphi_{i\beta}(x).$$
We will refer to this family of equations as incentive dynamics.

It should be clear that fixed points of this family of differential equations are exactly incentive equilibria. As a consequence we have a number of incentive dynamics we can already describe rather easily. First we note that if we allow the incentive to be given as Nash originally conceived, \( \varphi(x) = \varphi^N(x) \), then we recover the BNN equations as one would naturally expect.

We note that in the special case when \( \sum \varphi_{i\beta}(x) = 0 \) for every \( x \in \Delta \), the incentive dynamics reduce to simply \( \dot{x}_{ia} = \varphi_{ia}(x) \). The \( n \)-population replicator equations are given by

\[
\varphi^R_{ia}(x) = x_{ia}(u_i(e_\alpha, x_{-i}) - u_i(x))
\]

simply by recognizing that this incentive fits this special case. We have previously noted that there are many incentives that have the same equilibria as the replicator incentive above. These were given by \( \varphi^R_{ia}(x) = x_{ia}(u_i(e_\alpha, x_{-i}) + g_i(x)) \) where \( g(x) \) is an arbitrary function from \( \Delta \) to \( \mathbb{R}^n \). We derive the replicator equations as follows

\[
\dot{x}_{ia} = \varphi^R_{ia}(x) - x_{ia} \sum_{\beta} \varphi^R_{i\beta}(x)
= x_{ia}(u_i(e_\alpha, x_{-i}) + g_i(x)) - x_{ia} \sum_{\beta} x_{i\beta}(u_\beta(e_\beta, x_{-i}) + g_i(x))
= x_{ia}u_i(e_\alpha, x_{-i}) + x_{ia}g_i(x) - x_{ia} \sum_{\beta} x_{i\beta}u_\beta(e_\beta, x_{-i}) - x_{ia} \sum_{\beta} x_{i\beta}g_\beta(x)
= x_{ia}u_i(e_\alpha, x_{-i}) + x_{ia}g_i(x) - x_{ia}g_i(x) - x_{ia}u_i(x)
= x_{ia}(u_i(e_\alpha, x_{-i}) - u_i(x))
\]

Furthermore, we can recover all possible mean dynamics by defining the incentive

\[
\varphi^M_i(x) = \sum_{j \in S} x_{j} \rho_{ji}(u(x), x).
\]
The probability of switching from strategy $i$ to $j$ is given by $\rho_{ij}(u(x), x)/R$, where $R$ is constant. Thus $\sum_{j \in S} \rho_{ij}(u(x), x) = R$ for any $i$. Hence $\varphi^M(x)$ induces the mean dynamics as follows,

$$
\dot{x}_i = \varphi^M_i(x) - x_i \sum_{j \in S} \varphi^M_j(x)
$$

$$
= \sum_{j \in S} x_j \rho_{ji}(u(x), x) - x_i \sum_{j \in S} x_i \rho_{ij}(u(x), x)
$$

$$
= \sum_{j \in S} x_j \rho_{ji}(u(x), x) - x_i \sum_{j \in S} \rho_{ij}(u(x), x)
$$

$$
= \sum_{j \in S} x_j \rho_{ji}(u(x), x) - x_i \sum_{j \in S} \rho_{ij}(u(x), x)
$$

3.2 Generality

It has been mentioned that there are other dynamical system models for game play. We would then like to know if the model presented here is in fact fully general in the sense that we can achieve all possible game dynamics with an appropriate choice of incentive. In general a game dynamic will have the form

$$
\dot{x}_{ia} = F_{ia}(x)
$$

where we require $F_{ia}(x)$ to preserve the simplex. Therefore, we must have $\sum_{i} F_{ia}(x) = 0$ for every $x \in \Delta$ and $i \in N$. Also, we must have $F_{ia}(x) \geq 0$ if $x_{ia} = 0$. These conditions make $F(x)$ an incentive function by our definition and our incentive dynamics are exactly $\dot{x}_{ia} = F_{ia}(x)$. Therefore, we can recover any valid game dynamic by an appropriate choice of incentive. As noted above, the incentives that generate a specific dynamic need not be
unique.
Chapter 4

Asymptotic Stability and Incentive Stable States

4.1 Information Theory and The Replicator Dynamics

Information theory was originally developed by Claude Shannon and Warren Weaver [Sha01, SW49] as a mathematical framework to describe problems in communication including, but not limited to, data compression and storage. He introduced measures of information called entropy\(^1\). Shannon’s entropy, denoted \(H(P)\), is a measure of the average uncertainty in a random variable, \(P\). It can be interpreted as the average number of bits needed to encode a message drawn i.i.d. from \(P\). Maximizing the entropy can be used to give a lower bound on this average number of bits needed for encryption.

For our purposes, the concepts of cross entropy and relative entropy will be of great use. The Kullback-Leibler (KL divergence or \(D_{KL}\)) divergence [KL51] or relative entropy is a measure of information gain (loss) from one state to another. More precisely, it is an average measure of the additional bits needed to store \(y\) given a code optimized to store \(x\).

\(^1\)In fact, the Shannon entropy is simply the Boltzmann entropy [Jay65] without the constants.
It is defined as

\[
D_{KL}(x||y) = \sum_{\alpha} x_\alpha \ln \frac{x_\alpha}{y_\alpha}
= \sum_{\alpha} x_\alpha \ln x_\alpha - \sum_{\alpha} x_\alpha \ln y_\alpha
= H(x) - H(x, y).
\]

where \(H(x, y)\) is the cross entropy of \(x\) and \(y\). It should be clear that minimizing \(D_{KL}\) with respect to \(y\) is equivalent to minimizing the cross entropy term as well. Intuitively, this is trying to find the best distribution to approximate the ‘true’ distribution \(x\) and is well known as the Principle of Minimum Discrimination or Minimum Discrimination Information.

Recall the definition of an evolutionary stable state or ESS \([S^+74]\).

**Definition 4.1.** A strategy profile \(\hat{x} \in \Delta\) is an ESS if and only if \(u(\hat{x}, x) > u(x, x)\) for every \(x \neq \hat{x}\) in a neighborhood of \(\hat{x}\).

In this context there is a single population playing a symmetric game against itself. It has been shown by Harper \([Har11]\) that the KL divergence is a Lyapunov function for the replicator equation at an ESS\(^2\). This is equivalent to the Principle of Minimum Discrimination. Further connection between evolutionary games and information theory can be realized by expanding the KL divergence in a Taylor series along \(x = y\) and noting that the Hessian term is positive definite and is thus a metric. The derived metric, a localization of the global divergence, is called the Shahshahani metric \([Sha79]\) and it has been shown that the replicator dynamics are gradient flows of this metric \([HS98]\).

\(^2\)This result continues to be true for \(n\)-population games.
4.2 Incentive Stable States

The deep connections between information theory and the replicator dynamics lead us to believe that some of these properties are more general. Unfortunately, most of our incentives are not gradient flows of some Riemannian metric, but the Principle of Minimum Discrimination is compelling enough for us to believe we may be able to describe asymptotically stable states for the incentive dynamics. We begin by defining a notion of incentive stability that is closely related to the notion of ESS.

**Definition 4.2.** A strategy profile $\hat{x}$ is an incentive stable state or ISS if and only if

$$x_i \cdot \frac{\varphi_i(x)}{x_i} < \hat{x}_i \cdot \frac{\varphi_i(x)}{x_i}, \quad \forall i$$

for $x \neq \hat{x}$ in a neighborhood of $\hat{x}$.

The interpretation is exactly the same as in the ESS case: $\hat{x}$ is preferred to all distributions sufficiently close.

We can now show that all ISS are asymptotically stable for the corresponding incentive dynamics. Note: if there is only one agent we have a necessary and sufficient condition for the Kullback-Liebler divergence to be a strict Lyapunov function.

**Theorem 4.1.** If the state $\hat{x}$ is an interior incentive stable state for the corresponding incentive dynamics, then $\sum_i D_{KL}(\hat{x}_i || x_i)$ is a local Lyapunov function.
Proof. Define $V_i(x) = \KL(\hat{x}_i||x_i)$ and $V(x) = \sum_i V_i(x)$. Then we have the following:

$$\dot{V}_i(x) = -\sum_{\alpha} \frac{\dot{x}_{i\alpha}}{x_{i\alpha}^{\alpha}}$$

$$= -\sum_{\alpha} \frac{\dot{x}_{i\alpha}}{x_{i\alpha}^{\alpha}} \left[ \varphi_{i\alpha}(x) - x_{i\alpha} \sum_{\beta} \varphi_{i\beta}(x) \right]$$

$$= \sum_{\beta} \varphi_{i\beta}(x) \sum_{\alpha} \dot{x}_{i\alpha} - \sum_{\alpha} \frac{\dot{x}_{i\alpha}}{x_{i\alpha}^{\alpha}} \varphi_{i\alpha}(x)$$

$$= \sum_{\alpha} \frac{x_{i\alpha} - \dot{x}_{i\alpha}}{x_{i\alpha}^{\alpha}} \varphi_{i\alpha}(x) < 0$$

$\Leftrightarrow x_i \cdot \frac{\varphi_i(x)}{x_i} < \hat{x}_i \cdot \frac{\varphi_i(x)}{x_i}$

$\blacksquare$
Chapter 5

The Uniform Distribution

We recall incentive dynamics are given by

\[ \dot{x}_{ia} = \varphi_{ia}(x) - x_{ia} \sum_{\beta} \varphi_{i\beta}(x) \]

where \( \varphi(x) \) is the incentive for the game. We have proved that if the incentive for a finite game is continuous, there exists a fixed point characterized by

\[ \varphi_{ia}(\hat{x}) = \hat{x}_{ia} \sum_{\beta} \varphi_{i\beta}(\hat{x}) \forall \alpha, i. \]

Notice that if this occurs at the uniform distribution, either \( \varphi_{ia}(\hat{x}) \) are all zero, or they are all the same for each agent.

Nash’s original incentive function is fixed if and only if all the component incentives are zero and thus it can only be in the first case described above. In contrast, the incentive function given by \( \varphi^{P}_{ia}(x) = \sum_{\gamma} (a_{\alpha \gamma} - u_{i}(x))_+ \) is only zero when \( u_{i}(x) \geq \max_{\gamma} u_{i}(e_{\alpha}, e_{\gamma}) \), where \( e_{\gamma} \in S_{-i} \) which can occur at the uniform distribution only if the game is constant, which is a degenerate case of little interest. Despite their differences we will demonstrate that the two can agree under certain circumstances. Also, we will see that the latter incentive is globally asymptotically stable at a uniform Nash equilibrium where the canonical dynamics fail to converge.
5.0.1 A Bad Game of Rock-Paper-Scissors

The standard game of Rock-Paper-Scissors (RPS) is given as a two person zero sum game with payoffs given in the table below on the left.

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>-1, 1</th>
<th>1, -1</th>
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<tr>
<td>0, 0</td>
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<tr>
<td>-1, 1</td>
<td>0, 0</td>
<td>1, -1</td>
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<tr>
<td>1, -1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>

To the right of the RPS payoffs we have a generalized RPS with $a$ and $b$ both positive. The case when $b > a$, or an agent can lose more than it can win, is an important example of a game. The unique Nash equilibrium for this game is the uniform distribution. We have seen many examples of incentive dynamics that have Nash equilibrium as their interior fixed points, such as the replicator equations, projection dynamics, the logit equations, best reply dynamics, and the Brown-von Neumann-Nash equations. However, in every one of these cases the dynamics do not converge to the unique equilibrium as shown in the figures below\(^1\). This leads us to the natural question: does any incentive dynamic converge to a rest point from any initial point?

5.1 Agreement Among Incentives

We note the incentive $\varphi_{ia}^D(x) = \sum_\gamma (a_\alpha - u_i(x))_+$ can be rewritten in the form $\varphi_{ia}^D(x) = \sum_\gamma (u_i(e_\alpha, x_{-i}) - u_i(x) + a_\alpha - u_i(e_\alpha, x_{-i}))_+$, which shows that it is similar to a Nash comparison in that we are checking the payoff given the other agents’ strategies are fixed. However, we are tempering that comparison by taking away the amount by which the agent is not receiving a preferred payoff available in the game. We will now show there is a class of games, which includes general RPS, with the property that the uniform distribution is a

\(^1\)The images were produced using the Dynamo Mathematica package developed by Sandholm, Dokumaci, and Franchetti [SDF11]. Colors indicate speed: blue is slowest and red is fastest.
Nash equilibrium as well as an incentive equilibrium for $\varphi^D(x)$. First we will need the following lemma.

**Lemma 5.1.** If $A_i$ is the payoff matrix for the $i$th agent, then $\hat{x}$ is a Nash equilibrium where each agent is using the uniform distribution over its strategies if and only if for each $i$, $A_i$ has an equal sum across rows.

**Proof.** We begin by noting that for an interior Nash equilibrium we must have

$$u_i(e_1, x_{-i}) = \ldots = u_i(e_{s_i}, x_{-i}), \forall i.$$  

It should suffice then to calculate the value of just one of the $u_i(e_\alpha, x_{-i})$. We will use the $n$-linearity of the payoffs to complete the task.

$$u_1(e_1, x_{-1}) = \sum_{j_2=1}^{s_2} \frac{1}{s_2} u_1(e_1, e_{j_2}, x_3, \ldots, x_n)$$  

$$= \frac{1}{s_2} \sum_{j_2=1}^{s_2} u_1(e_1, e_{j_2}, x_3, \ldots, x_n)$$  

$$= \frac{1}{s_2 s_3} \sum_{j_2=1}^{s_2} \sum_{j_3=1}^{s_3} u_1(e_1, e_{j_2}, e_{j_3}, x_4, \ldots, x_n)$$  

$$= \ldots = \frac{1}{\prod_{i \in N/\{1\}} s_i} \sum_{j_2=1}^{s_2} \sum_{j_3=1}^{s_3} \cdots \sum_{j_n=1}^{s_n} u_1(e_1, e_{j_2}, e_{j_3}, \ldots, e_{j_n})$$  

$$= \frac{s_1}{|S|} \sum_{\beta} u_1(e_1, e_{-1}\beta)$$  

which is exactly the average of the coefficients in the first row of $A_1$. Thus for any agent $i$ we have the equalities

$$\frac{s_i}{|S|} \sum_{\beta} u_i(e_1, e_{-i}\beta) = \frac{s_i}{|S|} \sum_{\beta} u_i(e_2, e_{-i}\beta) = \ldots = \frac{s_i}{|S|} \sum_{\beta} u_i(e_{s_i}, e_{-i}\beta)$$

which after cancellation of the non-zero term $\frac{s_i}{|S|}$ proves our assertion. \qed
Proposition 5.2. If uniform distribution, \( \hat{x} \in \Delta \), is a Nash equilibrium and in each of the payoff matrices the sums of the elements in each row that are larger than the average are equal, then it is an incentive equilibrium for \( \varphi^D(x) \).

Proof. We will use the above lemma to prove the assertion. Given that the rows must all have an equal sum, the average of the elements in \( A_i \), which we will denote \( \bar{a}_i \), is equal to \( \frac{s_i}{|S|} \sum_{\beta} a_{i\beta} \). Let us now consider the condition for an incentive equilibrium when our incentive is given by \( \varphi^D(x) \). At a Nash equilibrium we have the following calculation for each agent \( i \)

\[
\varphi^D_{i\alpha}(\hat{x}) = \sum_{\gamma} (u_i(e_{i\alpha}, \hat{x}_{-i}) - u_i(\hat{x}) + a_{\alpha\gamma} - u_i(e_{i\alpha}, \hat{x}_{-i}))_+ \tag{5.6}
\]

\[
= \sum_{\gamma} (a_{\alpha\gamma} - u_i(e_{i\alpha}, \hat{x}_{-i}))_+ \tag{5.7}
\]

\[
= \sum_{\gamma} \left( a_{\alpha\gamma} - \frac{s_i}{|S|} \sum_{\beta} u_i(e_{i\alpha}, e_{-i\beta}) \right)_+ \tag{5.8}
\]

\[
= \sum_{\gamma} (a_{\alpha\gamma} - \bar{a}_i)_+ \tag{5.9}
\]

where the second line is justified since \( \hat{x} \) is a Nash equilibrium and thus \( u_i(e_{i\alpha}, \hat{x}_{-i}) = u_i(\hat{x}) \). The last line is simply the sum of all the elements from row \( \alpha \) that are larger than the average. Given our assumption, it must be the case that \( \varphi^D_{i\alpha}(\hat{x}) = \varphi^D_{i\beta}(\hat{x}) \) for every \( \alpha \) and \( \beta \). Thus we have \( \varphi^D_{i\alpha}(\hat{x}) = \frac{1}{s_i} \sum_{\beta} \varphi^D_{i\beta}(\hat{x}) \) for every agent \( i \), which is true if and only if \( \hat{x} \) is an incentive equilibrium.

To summarize, we found a class of games where the Nash equilibrium coincides with the incentive equilibrium for \( \varphi^D(x) \) at the uniform distribution. All RPS games have the property that the rows of the payoff matrices are permutations of the first row. Games with this property form a subset of the games where the Nash equilibrium and our incentive equilibrium agree.

We conjecture that this is the only agreement outside of constant games and strategies.
where players are receiving their respective maximum payoff. There are simple counterex-
amples when either of the conditions is dropped. For example, if $A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$, the average is 0, but the sums across rows are not equal. The interior Nash equilibrium is $\hat{x} = ((1/6, 1/6, 2/3), (1/6, 1/6, 2/3))$ while the incentive equilibrium for $\varphi^D$ is the uniform distribution. On the other hand, if $A_i = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ then the Nash equilibrium is the uniform distribution, but the incentive equilibrium is $\hat{x} \approx ((0.31, 0.69), (0.31, 0.69))$.

### 5.2 Asymptotic Stability

As we have seen, many of the dynamics that have Nash equilibria as fixed points do not necessarily converge to the uniform distribution. The specific examples that do (at least so far) have been Rock-Paper-Scissors type games. We notice that the main idea is to create a cycle of best replies by permuting the values in the first row of the payoff matrix. This cyclic behavior is essentially the problem with convergence. We will now show that changing the parameters while maintaining this type of cyclic payoff structure has no impact on the asymptotic stability of the incentive equilibrium for $\varphi^D(x)$.

**Proposition 5.3.** If the rows of the payoff matrix $A_i$ are permutations of each other, $\varphi_{ia}^D(x) = \varphi_{i\beta}^D(x)$ for all $x \in \Delta$ and either $\varphi_{ia}^D(\hat{x}) = 0$ or $\hat{x}_{ia} = \frac{1}{s_i}$ for every $\alpha$ at incentive equilibrium.

**Proof.** Denote $\sigma$ as the permutation that takes row $\alpha$ to row $\beta$; then every element in row $\beta$ can be written as $a_{i\beta\gamma} = a_{i\alpha\sigma(k)}$ for some $k \in S_i$. Thus $\varphi_{i\beta}^D = \sum_{\gamma}(a_{i\beta\gamma} - u_i(x))_+ = \sum_{k}(a_{i\alpha\sigma(k)} - u_i(x))_+ = \varphi_{ia}^D(x)$ regardless of $x$.

We can now use this fact to describe all possible incentive equilibria for $\varphi^D(x)$. By definition, at equilibrium $\hat{x}$, $\varphi_{ia}^D(\hat{x}) = \hat{x}_{ia} \sum_{\beta} \varphi_{i\beta}^D(\hat{x})$ for every $i$ and every $\alpha$. Given that the
incentive functions are all equal regardless of $x \in \Delta$, we must have $\varphi^D_{i\alpha}(\hat{x}) = \hat{x}_{i\alpha}s_i\varphi^D_{i\alpha}(\hat{x})$, which is true if and only if $\varphi^D_{i\alpha}(\hat{x}) = 0$ for all $\alpha$, which can occur only at the boundary or in a degenerate game, or when $\hat{x}_{i\alpha} = \frac{1}{s_i}$ for every $\alpha$.

Recall the definition of an ISS is

$$\hat{x}_i \cdot \frac{\varphi^D_i(x)}{x_i} > x_i \cdot \frac{\varphi^D_i(x)}{x_i}$$

for all $x \neq \hat{x}$ in some neighborhood of $\hat{x}$. Also, an ISS is asymptotically stable wherever it satisfies the inequality in the definition. It will suffice then to prove that the uniform distribution is an ISS and the entire space is its basin of attraction.

**Theorem 5.4.** If $\hat{x}$ is a uniform incentive equilibrium for $\varphi^D(x)$ and the payoff matrices have rows that are permutations of each other, then $\hat{x}$ is globally asymptotically stable in $\text{int}\Delta$ for the incentive dynamics.

**Proof.** The previous proposition gives us that the incentives are equal for all $\alpha$ so we can without loss of generality use only $\varphi^D_{i1}(x)$ for each $i$.

$$0 > -\frac{1}{s_i} \sum_{\alpha} \frac{\varphi^D_{i\alpha}(x)}{x_{i\alpha}} + \sum_{\beta} \varphi^D_{i\beta}(x)$$

$$= -\frac{\varphi^D_{i1}(x)}{s_i} \sum_{\alpha} \frac{1}{x_{i\alpha}} + s_i \varphi^D_{i1}(x)$$

$$= \frac{\varphi^D_{i1}(x)}{s_i} \left[ s_i^2 - \sum_{\alpha} \frac{1}{x_{i\alpha}} \right]$$

If we define $f(x) = \sum_{\alpha} \frac{1}{x_{i\alpha}}$ it is easy to show that $f(x)$ has a global minimum of $s_i^2$ when $x_i$ is the uniform distribution. We simply optimize using Lagrange multipliers, noting that the Hessian matrix of $f(x)$ is positive definite in the interior of $\Delta$. Thus $\hat{x}$ satisfies the ISS
definition for all $x \in \text{int}\Delta$.

We further conjecture that all interior incentive equilibrium are asymptotically stable. If this is true we can reduce the open problem of finding a game dynamic where every orbit converges to a rest point to proving that the basins of attraction for the incentive equilibrium form a partition of $\Delta$. 
Figure 5.1: Stable limit cycles in Bad RPS
Figure 5.2: The replicator dynamics display invariant limit cycles and unstable equilibrium points in the RPS

Figure 5.3: The projection dynamics display invariant limit cycles and unstable equilibrium points in the RPS
Figure 5.4: Global asymptotic stability of the uniform distribution in the simultaneous updating dynamics

(a) bad RPS
(b) RPS
(c) good RPS
Chapter 6

Numerical Results

Most of the incentives defined so far are closely related to the Nash equilibrium, which has been well studied. We would like to present some numerical results for the incentive based on simultaneous updating,

\[ \varphi^D_{i\alpha}(x) = \sum_{\gamma} (a_{i\alpha\gamma} - u_i(x))_+. \]

Two player games are by far the most researched in game theory and as such are the best place to start the discussion. We make the assumption that both players will be making decisions based on the incentive \( \varphi^D(x) \). Whenever possible experimental data will be given alongside our numerical approximations of behavior. Illustrations\(^1\) of the game dynamics are given whenever possible. Colors indicate speed: blue is slowest and red is fastest.

To calculate the equilibrium we will use several consequences of the definition for incentive equilibrium. First we note that if \( \hat{x}_{i\alpha} = 0 \), then the definition can only be satisfied if \( \varphi_{i\alpha}(x) = 0 \) as well. This can only occur for \( \varphi^D_{i\alpha}(x) \) if and only if the equilibrium utility

\[ u_i(\hat{x}) \geq \max_{\gamma} u_i(e_{i\alpha}, e_{-i,\gamma}). \]

Consequently, if \( \hat{x} \in S \), then \( u_i(\hat{x}) \) must be the maximum possible payoff for each agent \( i \). Given these strict conditions for pure strategy equilibrium, we expect most games will have an equilibrium point in the interior of \( \Delta \). If this is the case, the incentive equilibrium can

\(^1\)All diagrams have been produced with Sandholm, Dokumaci, and Franchetti’s Dynamo Mathematica package [SDF11]
be classified by the following system of equations:

$$\sum_{\beta} \varphi_{i\beta}(\hat{x}) = \frac{\varphi_{i1}(\hat{x})}{\hat{x}_{i1}} = \frac{\varphi_{i2}(\hat{x})}{\hat{x}_{i2}} = \cdots = \frac{\varphi_{is_i}(\hat{x})}{\hat{x}_{is_i}}.$$ 

Given our proposition on symmetric games given in chapter 2, we can further simplify calculations by finding the incentive equilibrium guaranteed in the closed convex subset of $\Delta$ given by the symmetry. For example, many of our games will have the form $u_1(x_1, x_2) = u_2(x_2, x_1)$, where along $U = \{x|x_1 = x_2\}$, $\varphi_{11}^D(x) = \varphi_{21}^D(x)$. Thus an equilibrium exists in $U$ and we have reduced our calculation to just player 1.

**6.0.1 Human Behavior**

One of the main criticisms of the Nash equilibrium is the general lack of adherence amongst agents to the equilibrium in game play by humans [GR72, RC65, TS04]. We will show many examples of this below with specific games. Specifically, we will see a number of examples where the Nash equilibrium is not only unique but also consists of only pure strategies. This is a very strong prediction as it can be interpreted as all rational actors must choose the prescribed pure strategy one hundred percent of the time. In practice this is somewhat, and not surprisingly, false. It is most commonly described as irrational behavior by agents.

Then there are the dilemmas; see Prisoner’s Dilemma and Traveler’s Dilemma below, where the equilibrium point itself seems to be questionable given what some might refer to as common sense. The Traveler’s Dilemma is particularly distressing as the equilibrium point seems to be diametrically opposed to the intuitive selection of choices for the game. In fact, in experiments [Bas94, Bas07, CCG07] very few, if any, agents played their Nash equilibrium strategy. Despite being a simpler game, the Prisoner’s Dilemma has similar, but less severe, observed behavior. Even in trials with experts playing the game once against each other, nearly one third played the strictly dominated strategy in the game.
Cardinal vs. Ordinal Dependence

The term ordinal dependence is used to describe the situation where agents are only concerned with the ranking of payoffs rather than absolute differences in utility. Applying a monotone increasing function to utilities does not affect ranking and it can be easily shown that it also does not affect the location of pure strategy Nash equilibria\(^2\). Thus any game with a unique pure strategy Nash equilibrium will have the same game play as any other game that can be obtained by an application of a monotone increasing function. Two such examples are the Dilemmas discussed above. However, in practice human actors in trials have shown a strong tendency towards cardinal dependence [GR72, CCG07]. If absolute differences are important as well as ranking then we expect the only way to achieve identical behavior after modifying a game is by a simple translation.

6.1 2x2 Games

6.1.1 Prisoner’s Dilemma

The much studied Prisoner’s Dilemma

\[
\begin{array}{c|cc}
 & \text{Confess} & \text{Not Confess} \\
\hline
\text{Confess} & P, P & T, S \\
\text{Not Confess} & S, T & R, R \\
\end{array}
\]

where \( T > R > P > S \) is given here in its ordinal form. \( R \) is the reward for cooperation if both prisoners choose not to confess to some crime, \( T \) is the temptation to confess, \( S \) is the sucker’s payoff for the agent that does not confess when the other does, and \( P \) is the punishment for mutual confession. Notice that with this ordering, confessing is a strictly dominating strategy for both the row and column players. Thus the Nash equilibrium is

\(^2\)Nash equilibria that contain mixed strategies can be affected by the application of a monotone increasing function.
always (confess, confess) or \(((1,0),(1,0))\) where the first coordinate is the probability of confessing. Suppose we use \(S = 1, P = 2, R = 3,\) and \(T = 4\) to represent the ordinal game. Then we have below the phase diagrams for the simultaneous updating and BNN. The point that appears to be asymptotically stable in the simultaneous updating portrait is \(x = ((1/\sqrt{2},1-1/\sqrt{2}),(1/\sqrt{2},1-1/\sqrt{2})) \approx ((.70, .30), (.70, .30))\) which is easily computable with the assumptions that \(2 < u_i(x) < 3\) and \(x_1 = x_2.\)

However, as we mentioned, the incentive equilibrium will have a strong dependence on the actual payoffs. If we change the temptation payoff to \(T = 10\) for example the equilibrium point is

\[
x_{11} = \frac{1}{18} \left(8 - \frac{44}{(917 + 9\sqrt{11433})^{1/3}} + (917 + 9\sqrt{11433})^{1/3}\right) \approx 0.932
\]

which is closer to the Nash equilibrium for the game, which does not change.

\(\epsilon\)-Prisoner’s Dilemma

To further illustrate the strong dependence on actual payoffs we define the \(\epsilon\)-Prisoner’s Dilemma by the following table for any \(\epsilon > 0.\)

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>1 + (\epsilon), (-\epsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\epsilon), (1 + \epsilon)</td>
<td>1, 1</td>
<td></td>
</tr>
</tbody>
</table>

We first notice that if \(\epsilon = 1\) and we translate the payoffs up by 2 we achieve the ordinal Prisoner’s Dilemma from above. Also, we have the same strict dominance of the second strategy by the first for any \(\epsilon > 0\) and we maintain the same symmetry as above. A simple calculation gives \(u_1(x) = 1 - x_{11}\) for all \(x \in U,\) and if we assume at equilibrium \(0 < u_1(\hat{x}) < 1\) our equilibrium occurs when

\[
\frac{\epsilon + \hat{x}_{11}}{\hat{x}_{11}} = \frac{\hat{x}_{11}}{1 - \hat{x}_{11}}.
\]
Figure 6.1: Ordinal Prisoner’s Dilemma.

Figure 6.2: Simultaneous updating for the Prisoner’s Dilemma with $T = 10$. 
Thus our solution is \( \hat{x}_{11} = \frac{1 - \epsilon + \sqrt{1 + 6\epsilon + \epsilon^2}}{4} \). A simple analysis of \( \hat{x}_{11} \) shows that it is a (strictly) increasing function of \( \epsilon \) and consequently \( \hat{x}_{11}(0) = \frac{1}{2} < \hat{x}_{11}(\epsilon) < 1 = \lim_{\epsilon \to \infty} \hat{x}_{11}(\epsilon) \).

Thus the equilibrium utility, \( 0 < u_1(\hat{x}) < \frac{1}{2} \), satisfies our assumption for all \( \epsilon > 0 \).

6.1.2 Chicken

The game of Chicken, or Hawk-Dove, is also of interest to game theorists and biologists. The ordinal payoff matrix is given below with the interpretation that each agent prefers a win to a tie, a tie to a loss, and a loss to a crash.

<table>
<thead>
<tr>
<th></th>
<th>Swerve</th>
<th>Straight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swerve</td>
<td>Tie , Tie</td>
<td>Lose , Win</td>
</tr>
<tr>
<td>Straight</td>
<td>Win , Lose</td>
<td>Crash , Crash</td>
</tr>
</tbody>
</table>

There are 3 Nash equilibria for this game. The two pure strategy equilibria can be described without committing any numerical values to the possible events. They are \( x = ((1,0),(0,1)) \) and \( x = ((0,1),(1,0)) \). There is also a mixed strategy equilibrium which depends on the actual payoffs. No such pure strategy incentive equilibrium exists as there is no win-win in the game. We have 3 variations on the ordinal game of chicken.

\[
\begin{array}{ccc}
3,3 & 2,5 & 0,0 \\
5,2 & 1,1 & 2,7 \\
& 6,6 & 1,-1 \\
& -10,-10 & -1,1 \\
\end{array}
\]

In the first two games the mixed Nash equilibrium is the same, \( x = ((1/3, 2/3), (1/3, 2/3)) \) with utility \( u(x) = 7/3 \) and \( u(x) = 14/3 \). The incentive equilibria is roughly \( x \approx (.24, .76), (.24, .76) \) with \( u(x) \approx 2.04 \) and \( x \approx (.58, .42), (.58, .42) \) with \( u(x) \approx 3.27 \) respectively. The final game displays a much wider difference in styles as the Nash equilibrium is \( x = ((9/10, 1/10), (9/10, 1/10)) \) with \( u(x) = -1/10 \), where the incentive equilibrium is \( x \approx (0.52, 0.48), (0.52, 0.48) \) with \( u(x) \approx -2.29 \).
Figure 6.3: Variations of Chicken in the simultaneous updating protocol
Figure 6.4: Variations of Chicken in the BNN

(a) Variation 1

(b) Variation 2

(c) Variation 3
6.1.3 Matching Pennies

Matching Pennies is a game designed to have cyclic behavior. That is, suppose we have the payoff matrix below, then the row player wins when it guesses which side of the coin the column player has facing up. If the column player thinks the row player knows what it has done, it is in the player’s best interest to switch. But then it is in the row player’s best interest to switch and processes of switching and following continues ad infinitum.

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Games of this nature are easy to analyze as both the Nash equilibrium and the incentive equilibrium are the uniform distribution, $x = ((1/2, 1/2), (1/2, 1/2))$ since the rows of the agent’s payoff matrices consists of permutations of the first row only.

An Asymmetric Variant of Matching Pennies

This asymmetric version of the Matching Pennies game puts the row player at an advantage. There are 3 possibilities for row to win in contrast to columns 2 chances.

<table>
<thead>
<tr>
<th></th>
<th>1, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

There are two Nash equilibria in this game at $x = ((1, 0), (0, 1))$ and $x = ((1, 0), (1/2, 1/2))$ that are also incentive equilibria for the game. There is an additional incentive equilibrium located at $x = ((2/3, 1/3), (1/2, 1/2))$. The main differences are that if $x_{21} = 1$ than any profile with $x_{11} > 1/2$ is a Nash equilibrium. Also, note in the figures below that the BNN has at best stable equilibria, where the simultaneous updating has two asymptotically stable fixed points. If one restricts to the boundary $x_{11} = 1$ then the win-win is not stable.
The others are asymptotically stable when restricted to the boundary and the interior of $\Delta$ respectively. This behavior is rather competitive and has been observed in human actors by Guyer and Rapoport [GR72] using other games. However, the conclusion is that the advantaged player seeks to find a larger margin of victory rather than settle for a tie.

![Figure 6.5: Asymmetric Matching Pennies](image)

(a) Simultaneous Updating  
(b) BNN

Figure 6.5: Asymmetric Matching Pennies

### 6.1.4 Coordination Games

As the name suggests, these games have the property that it is best for agents to choose the same strategy. They all have the same pure strategy Nash equilibria at $x = ((1, 0), (1, 0))$ and $x = ((0, 1), (0, 1))$. Again the mixed strategy equilibrium will depend on the actual payoffs if it exists.

<table>
<thead>
<tr>
<th></th>
<th>$A, a$</th>
<th>$B, b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C, c$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D, d$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

48
where \( A > B, \ D > C \) for player 1 and \( a > c, \ d > b \) for player 2. We will present a few of the more common examples of coordination games.

**Battle of the Sexes**

This game is supposed to model a couple who prefer to spend time together but have differing interests. A common way of describing the situation is that strategy 1 is to go to a football game and strategy 2 is to go to the opera. Row prefers football while column prefers opera.

<table>
<thead>
<tr>
<th>Opera</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>2, 1</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

The equilibrium points in this game are very similar. The mixed Nash equilibrium is \( x = ((2/3, 1/3), (1/3, 2/3)) \) versus \( x \approx ((0.76, 0.24), (0.24, 0.76)) \) for the incentive equilibrium. In this example we use the symmetry \( U = \{ x \in \Delta | x_{11} = x_{22} \} \).

![Figure 6.6: Battle of the Sexes](image)
Stag Hunt

The Stag Hunt is another coordination game where players have the choice to hunt a deer and possibly catch it so there is enough food for both or to individually catch a rabbit for sure.

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>10,10</td>
<td>0,7</td>
</tr>
<tr>
<td>Hare</td>
<td>7,0</td>
<td>7,7</td>
</tr>
</tbody>
</table>

This game’s dynamics are somewhat trivial for the simultaneous updating. The players only incentive equilibrium is $x = ((1,0),(1,0))$ which is also Nash. However, there is a mixed Nash at $x = ((7/10,3/10),(7/10,3/10))$ which would change if the payoffs were altered. The incentive equilibrium however will not change and will always fix on the win-win with this type of configuration.

Figure 6.7: Stag Hunt

(a) Simultaneous Updating  (b) BNN
6.1.5 Other Games

Guyer and Rapoport conducted a study of $2 \times 2$ games which they summarize in [GR72]. We will present their findings alongside our incentive equilibrium. Their are 6 classes of games and several variations for each one. Some of our results are close to the actual frequencies seen in the study while others are not.

For symmetric games we will only give the first coordinate $x_{11} = p$ since all other coordinates will depend only on $p$.

No Conflict

There are 3 variations of the No Conflict games. They are given below.

\[
\begin{array}{|c|c|c|c|}
\hline
28, 28 & 2, 20 & 28, 28 & 12, 20 \\
20, 2 & -10, -10 & 20, 12 & -10, -10 \\
\hline
\end{array}
\]

The games are symmetric and have a win-win. It turns out that in all three cases our only incentive equilibrium is $p = 1$ which is also the Nash equilibrium. The experimental data is $p = 0.88$, $p = 0.91$, and $p = 0.91$ respectively. The agents seemed to play strategy two more as the possibility of a larger margin of victory increased.

Threat Vulnerable

There are 7 variations given for this type of game.

\[
\begin{array}{|c|c|c|c|c|}
\hline
-1, 20 & 20, 15 & 8, 20 & 20, 15 \\
-2, 6 & 15, -2 & -2, 6 & 15, -2 \\
\hline
8, 20 & 20, 15 & 8, 20 & 20, 15 \\
-8, 6 & 15, -2 & -8, 14 & 15, -2 \\
\hline
\end{array}
\]

Our equilibrium will be given as $(x_{11}, x_{21}) = (p, q)$ since the remaining two components
depend on these two.

<table>
<thead>
<tr>
<th>Variation</th>
<th>Incentive Equilibrium</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.58, 0.76)</td>
<td>(0.86, 0.84)</td>
</tr>
<tr>
<td>2</td>
<td>(0.64, 0.89)</td>
<td>(0.92, 0.87)</td>
</tr>
<tr>
<td>3</td>
<td>(0.73, 1)</td>
<td>(0.93, 0.89)</td>
</tr>
<tr>
<td>4</td>
<td>(0.64, 0.70)</td>
<td>(0.92, 0.81)</td>
</tr>
<tr>
<td>5</td>
<td>(0.65, 1)</td>
<td>(0.94, 0.87)</td>
</tr>
<tr>
<td>6</td>
<td>(0.65, 1)</td>
<td>(0.93, 0.88)</td>
</tr>
<tr>
<td>7</td>
<td>(0.65, 1)</td>
<td>(0.95, 0.86)</td>
</tr>
</tbody>
</table>

As the Nash equilibrium in all of these games is (1, 1) we have done a better job of predicting the column player but worse for row.

**Force Vulnerable**

The Force Vulnerable game is essentially an asymmetric Prisoner’s Dilemma. We again have 7 variants all with Nash equilibrium (1, 1).
These games are very similar to the Battle of the Sexes and are in fact coordination games. There are 5 variations. As with the other coordination games these games have 2 pure strategy Nash equilibria and one mixed Nash.

Again, we have done better approximating the column player than the row player.
Leader

Leader games are symmetric games where the players would rather not choose the same strategy, which is similar to the concept in Chicken. There are two pure strategy Nash equilibria located at $(1, 0)$ and $(0, 1)$. There is also a mixed strategy equilibrium.

<table>
<thead>
<tr>
<th>12, 12</th>
<th>15, 32</th>
<th>12, 12</th>
<th>15, 21</th>
<th>12, 12</th>
<th>15, 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>32, 15</td>
<td>−5, −5</td>
<td>21, 15</td>
<td>−5, −5</td>
<td>16, 15</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>

Variation | Incentive | Equilibrium | Observations | Mixed Nash |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30</td>
<td>0.68</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.42</td>
<td>0.77</td>
<td>20/29</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.54</td>
<td>0.83</td>
<td>5/6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.45</td>
<td>0.77</td>
<td>26/35</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.48</td>
<td>0.79</td>
<td>31/40</td>
<td></td>
</tr>
</tbody>
</table>

The mixed strategy Nash is actually better than our approximation in every case.

Hero

Hero games are much like Leader games. The difference is the leader is increasing their own payoff but increasing the other’s payoff more.

<table>
<thead>
<tr>
<th>8, 8</th>
<th>16, 14</th>
<th>8, 8</th>
<th>24, 14</th>
<th>8, 8</th>
<th>30, 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>14, 16</td>
<td>−5, −5</td>
<td>14, 16</td>
<td>−5, −5</td>
<td>14, 30</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8, 8</th>
<th>24, 22</th>
<th>8, 8</th>
<th>24, 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>14, 16</td>
<td>−5, −5</td>
<td>9, 24</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>
Cycle games are roughly similar to Matching Pennies in the sense that during repeated trials there is always incentive to change one’s strategy. There is a single pure strategy Nash equilibrium at (1, 1) and a mixed strategy Nash equilibrium.

<table>
<thead>
<tr>
<th>Variation</th>
<th>Incentive Equilibrium</th>
<th>Observations</th>
<th>Mixed Nash</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.59</td>
<td>0.84</td>
<td>7/9</td>
</tr>
<tr>
<td>2</td>
<td>0.81</td>
<td>0.83</td>
<td>29/35</td>
</tr>
<tr>
<td>3</td>
<td>0.86</td>
<td>0.82</td>
<td>35/41</td>
</tr>
<tr>
<td>4</td>
<td>0.55</td>
<td>0.76</td>
<td>29/43</td>
</tr>
<tr>
<td>5</td>
<td>0.97</td>
<td>0.91</td>
<td>29/30</td>
</tr>
</tbody>
</table>
6.2 3x3 and Larger Games

We have already discussed one class of $3 \times 3$ and larger games when discussing the uniform distribution. If the rows of the payoff matrix are permutations of the first row then the uniform distribution is an incentive equilibrium and it is asymptotically stable for the entire interior of $\Delta$. We will now switch focus to a large game of interest.

6.2.1 Traveler’s Dilemma

The Traveler’s Dilemma is a two agent game with the following rules. Both agents must offer a price for identical objects obtained on vacation that have been broken while in transit. The objects are identical, but the person who claims the lower value gets that value plus a reward. The other player gets the lower bid minus the reward. If they both say the same price, they both receive that value. The game was originally introduced by Basu [Bas94, Bas07] and has since been studied by many, including [CCG07, CGGH99, GH99] as well as others. The dilemma is that the Nash equilibrium for this game is always the minimum possible bid for the item. Basu’s original formulation had this minimum at 2 with a reward/punishment of $r = 2$. Others have noted in their experiments that increasing $r$ will increase the likelihood of agents playing Nash strategy. However, for lower $r$, like that found in Basu’s study, the Nash strategy is virtually if not entirely non-existent in actual game play.

<table>
<thead>
<tr>
<th>Bid</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2,2</td>
<td>4,0</td>
<td>4,0</td>
</tr>
<tr>
<td>3</td>
<td>0,4</td>
<td>3,3</td>
<td>5,1</td>
</tr>
<tr>
<td>4</td>
<td>0,4</td>
<td>1,5</td>
<td>4,4</td>
</tr>
</tbody>
</table>

For the $3 \times 3$ version of the Traveler’s Dilemma above the incentive equilibrium is very
easy to compute. In fact, if we assume $2 < u < 3$, our calculation becomes

$$\frac{2(4 - u)}{x_{11}} = \frac{5 - u + 3 - u}{x_{12}} = \frac{4 - u}{x_{13}}$$

which clearly gives $x_{11} = x_{12} = 2x_{13}$. The simplex constraint is enough to finish the calculation giving $x = ((2/5, 2/5, 1/5), (2/5, 2/5, 1/5))$ with a payoff of $u(x) = 2.4$. 

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Figure 6.8: $3 \times 3$ Traveler's Dilemma
(a) Best Response

(b) Smith

(c) logit $\mu = 0.2$

Figure 6.9: $3 \times 3$ Traveler's Dilemma
References


