ANALYSIS OF TRIANGLE QUALITY MEASURES

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Abstract. Several of the more commonly used triangle quality measures are analyzed and compared. Proofs are provided to verify that they do exhibit the expected extremal properties. The asymptotic behavior of these measures is investigated and a number of useful results are derived. It is shown that some of the quality measures are equivalent, in the sense of displaying the same extremal and asymptotic behavior, and that it is therefore possible to achieve a concise classification of triangle quality measures.

Introduction

Assessment of mesh quality is an important requirement in both the selection of a finite element mesh and the evaluation of meshes that have undergone adaptation [2, 5]. Several measures of element quality have been proposed [1, 8, 11] based on the dimensionless ratios of various geometric parameters. Apart from the work of [11], there appears to be almost no discussion in the literature on the relative merits of these particular quality measures. More recently, alternative quality measures have been suggested [3, 7, 9, 10]. These alternative measures are derived from the singular values of a matrix whose columns are formed by the edge vectors of the mesh element.

An element is said to be degenerate if its volume is zero. Let \( Q \) be a quality measure defined for any non-degenerate simplex \( t \) and let the range of \( Q \) be the real interval \([1, +\infty]\. It is assumed that \( Q \) satisfies the following extremal properties,

(i) \( Q \) attains its minimum value of \( 1 \) if and only if \( t \) is a regular simplex.
(ii) \( Q \) has no other extrema.

In many cases, these extremal properties have been assumed, or stated, without proof. Although the extremal behavior of these quality measures might appear obvious, we believe that this behavior should be established rigorously and precise bounds should be found.

In this paper we examine the triangular case since properties of the triangle are particularly amenable to analysis. We consider several of the more commonly used triangle quality measures, provide proofs of their extremal properties and examine their asymptotic behavior. Our goal is to provide a number of useful results on triangle quality measures that may lead to a better assessment of both planar triangulations and triangulated surfaces.

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1. Preliminaries

In this paper, we consider a non-degenerate triangle $t = ABC$ with area $A$, half-perimeter $p$, edges of lengths $a = BC$, $b = AC$ and $c = AB$, and denote the angle at vertex $A$ (resp. $B$, $C$) as $\alpha$ (resp. $\beta$, $\gamma$) and the radius of the inscribed (resp. circumscribed) circle of $t$ as $r$ (resp. $R$). In addition, the vertices $A$, $B$ and $C$ are defined respectively by the position vectors $v_0$, $v_1$ and $v_2$ in an arbitrary orthonormal affine reference frame. For simplicity, we choose a frame of reference parallel to the plane of the triangle $t$, in which case the coordinates of the position vectors $v_0$, $v_1$ and $v_2$ are respectively denoted as $(x_0, y_0)$, $(x_1, y_1)$ and $(x_2, y_2)$.

We shall also use the following standard norm-like notations:

$$
||t||_0 = \min(a, b, c) \\
||t||_2 = \sqrt{a^2 + b^2 + c^2} \\
||t||_\infty = \max(a, b, c) \\
\theta_0 = \min(\alpha, \beta, \gamma) \\
\theta_\infty = \max(\alpha, \beta, \gamma).
$$

We shall assume without proof a number of results from elementary geometry (see for example [6] for proofs and details). In particular, we will make use of the following well-known relations:

$$
2R = \frac{abc}{2A} = \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma},
$$

where $A$ is given by

$$
A = rp,
$$

as well as by Heron’s formula:

$$
A = \sqrt{(p-a)(p-b)(p-c)}.
$$

We also recall two important results, valid for any $n \in \mathbb{N}^*$, the arithmetic-geometric inequality

$$
(\forall (u_1, \ldots, u_n) \in \mathbb{R}_+^n) \quad \sqrt[n]{\prod_{k=1}^{k=n} u_k} \leq \frac{1}{n} \sum_{k=1}^{k=n} u_k
$$

and Cauchy-Schwarz inequality

$$
(\forall (u_1, \ldots, u_n) \in \mathbb{R}_+^n) \quad \sum_{k=1}^{k=n} u_k \leq \sqrt{n} \sqrt{\sum_{k=1}^{k=n} u_k^2}.
$$

We shall use the terms needle and flattened triangle to refer to two types of nearly degenerate triangles. More precisely, these are defined as follows.

**Definition 1.1.** A needle is a non-degenerate triangle that has one and only one angle close to 0.

**Definition 1.2.** A flattened triangle is a non-degenerate triangle that has one angle close to $\pi$. 
2. Extremal angles

One of the most commonly accepted means of measuring triangle quality is to examine $\theta_0$ or $\theta_\infty$. By definition of these angles, one always has:

\begin{equation}
0 < \theta_0 \leq \frac{\pi}{3} \leq \theta_\infty.
\end{equation}

Moreover, the implicit condition

\begin{equation}
\theta_0 \leq \pi - \theta_0 - \theta_\infty \leq \theta_\infty
\end{equation}

is equivalent to the following:

\begin{equation}
\theta_\infty \leq \pi - 2\theta_0, \quad 2\theta_\infty \geq \pi - \theta_0.
\end{equation}

For the sake of clarity, (2.1) and (2.3) are summarized by denoting as $\Omega$ the corresponding definition set of $(\theta_0, \theta_\infty)$, shown in figure 1; the bold edges correspond to the two different kinds of isosceles triangle: acute on the lower one, obtuse on the upper one, which meet when $\theta_0 = \theta_\infty = \frac{\pi}{3}$ (i.e. the equilateral case).

![Figure 1](image-url)

Figure 1. Gray: $\Omega$, definition set of $\theta_0$ and $\theta_\infty$; the thin edge with its endpoints are excluded.

In other words, (2.3) gives the bounds for the maximal angle depending on the minimal one:

\begin{equation}
(\forall \theta_0 \in \left[0, \frac{\pi}{3}\right]) \quad \frac{\pi}{2} - \frac{\theta_0}{2} \leq \theta_\infty \leq \pi - 2\theta_0
\end{equation}

while (2.1) and (2.2) conversely provide:

\begin{equation}
(\forall \theta_\infty \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]) \quad \pi - 2\theta_\infty \leq \theta_0 \leq \frac{\pi}{2} - \frac{\theta_\infty}{2}
\end{equation}

\begin{equation}
(\forall \theta_\infty \in \left[\frac{\pi}{2}, \pi\right]) \quad 0 < \theta_0 \leq \frac{\pi}{2} - \frac{\theta_\infty}{2}
\end{equation}

Two useful inequalities directly arise, from (2.4):

\begin{equation}
(\forall (\theta_0, \theta_\infty) \in \Omega) \quad \frac{\pi}{2} \leq \frac{\theta_0}{2} + \theta_\infty \leq \pi - \frac{\theta_0}{2}
\end{equation}

and from (2.5) and (2.6):

\begin{equation}
(\forall (\theta_0, \theta_\infty) \in \Omega) \quad 0 < \max \left(\frac{\theta_0}{2}, \pi - \frac{\theta_0}{2}\right) \leq \frac{\theta_0}{2} + \theta_0 \leq \frac{\pi}{2}.
\end{equation}
These inequalities may be rewritten strictly over the interior of $\Omega$ of $\Omega$: 

$$\left( \forall (\theta_0, \theta_\infty) \in \overset{\circ}{\Omega} \right) \quad \frac{\theta_0}{2} < \frac{\theta_\infty}{2} + \theta_\infty < \pi - \frac{\theta_0}{2}$$

and

$$\left( \forall (\theta_0, \theta_\infty) \in \overset{\circ}{\Omega} \right) \quad 0 < \max \left( \frac{\theta_0}{2}, \pi - \frac{3\theta_\infty}{2} \right) < \frac{\theta_\infty}{2} + \theta_0 < \frac{\pi}{2}.$$ 

Before examining how the more commonly used non-dimensional quality measures are related to these extremal angles, it is useful to prove the following result. It seems obvious but we have not observed it in literature.

**Lemma 2.1.** The three angles of a non-degenerate triangle are sorted in the same order as the lengths of their respective opposite edges.

**Proof.** We can assume, without loss of generality, that $a \leq b \leq c$. Hence, it can be directly deduced from (1.1) that $\sin \alpha \leq \sin \beta \leq \sin \gamma$. Now, both $\alpha$ and $\beta$ are necessarily smaller than $\frac{\pi}{2}$ and the sine function is monotonically increasing over $[0, \frac{\pi}{2}]$, thus $\alpha \leq \beta$. Concerning $\gamma$, two cases may occur: if it is acute, then the same argument can be used; if it is obtuse or right, then clearly $\beta \leq \gamma$. Therefore, in either case, we have $\alpha \leq \beta \leq \gamma$. \hfill $\square$

**Remark 2.2.** In other words, the length of the edge opposite angle $\theta_0$ (resp. $\theta_\infty$) is $|\ell_0|$ (resp. $|\ell_\infty|$).

## 3. Radius-Ratio

A convenient, non-dimensional and thus homogeneous, quality measure consists in comparing $r$ and $R$, resulting in the *radius-ratio*, defined as:

$$\rho = \frac{R}{r}. \quad (3.1)$$

Combining (1.1) with (1.2) leads to:

$$abc = 4Rrp \quad (3.2)$$

and then a further application of (1.1) gives the following expressions for $\rho$ in terms of the angles of a triangle:

$$\rho = \frac{\sin \alpha + \sin \beta + \sin \gamma}{2 \sin \alpha \sin \beta \sin \gamma} \quad (3.3)$$

whence, since $\alpha + \beta + \gamma = \pi$,

$$\rho = \frac{\sin \alpha + \sin \beta + \sin(\alpha + \beta)}{2 \sin \alpha \sin \beta \sin(\alpha + \beta)} \quad (3.4)$$

or, using the extremal angles,

$$\rho = \frac{\sin \theta_0 + \sin \theta_\infty + \sin(\theta_0 + \theta_\infty)}{2 \sin \theta_0 \sin \theta_\infty \sin(\theta_0 + \theta_\infty)} \quad (3.5)$$

If we write $p = \tan \frac{\theta_0}{2}$ and $q = \tan \frac{\theta_\infty}{2}$ in (3.4), then

$$\rho = \frac{(1 + p^2)(1 + q^2)}{4pq(1 - pq)} \quad (3.6)$$
3.1. **Extremum.** The mapping \( x \mapsto \tan \frac{x}{2} \) is a \( C^\infty \) function, bijective from \([0, \pi]\) onto \( \mathbb{R}^*_+ \). Thus both parametrizations of \( \rho \) are \( C^\infty \)-equivalent and, hence, differential properties are bijectively transported from the one to the other. Now, it is easy to prove that the condition

\[
(\alpha, \beta) \in [0, \pi]^2, \quad \alpha + \beta < \pi
\]

is equivalent to

\[
(p, q) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+, \quad pq < 1.
\]

Therefore, let us consider the mapping

\[
f : \mathcal{D} \rightarrow \mathbb{R}^*_+ \times \mathbb{R}^*_+ \quad \quad (p, q) \mapsto \frac{(1 + p^2)(1 + q^2)}{pq(1 - pq)}
\]

where \( \mathcal{D} \) is the domain\(^1 \) defined by condition (3.8). Clearly, \( f \) is \( C^\infty \), when \( \mathcal{D} \) is open; hence, any extremum of \( f \) is attained at a stationary point. We have:

\[
\frac{\partial f}{\partial p}(p, q) = \frac{(1 + q^2)(p^2 + 2pq - 1)}{pq^2(1 - pq)^2}
\]

\[
\frac{\partial f}{\partial q}(p, q) = \frac{(1 + p^2)(q^2 + 2pq - 1)}{pq^2(1 - pq)^2}
\]

and \((p, q)\) is a stationary point if and only if

\[
p^2 + 2pq - 1 = q^2 + 2pq - 1 = 0.
\]

Hence, this implies \( p = q \) and, since we know that the variables substitution is bijective, \( \alpha = \beta \). Now, assuming this necessary condition is satisfied, (3.12) becomes \( 3p^2 = 1 \), so that \( p = q = \frac{1}{\sqrt{3}} \) since these values must be positive. In other words, the only critical point of \( \rho \) is met when \( \alpha = \beta = \frac{\pi}{6} \) (i.e. for an equilateral triangle). In order to check whether this case corresponds, as expected, to a minimum, one has to make sure that the hessian matrix is positive definite. The second-order derivatives are given by:

\[
\frac{\partial^2 f}{\partial p^2}(p, q) = 2 \frac{(1 + q^2)(p^2q + 3p^2q^2 - 3pq + 1)}{pq^2(1 - pq)^3}
\]

\[
\frac{\partial^2 f}{\partial q^2}(p, q) = 2 \frac{(1 + p^2)(q^2p + 3p^2q^2 - 3pq + 1)}{pq^2(1 - pq)^3}
\]

\[
\frac{\partial^2 f}{\partial p \partial q}(p, q) = \frac{pq + 5}{(1 - pq)^3} + \frac{(p^2q^2 - 1)(3pq - 1)}{p^2q^2(1 - pq)^3}
\]

which gives, when \( p = q = \frac{1}{\sqrt{3}} \):

\[
\frac{\partial^2 f}{\partial p^2} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \frac{\partial^2 f}{\partial q^2} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = 36
\]

\[
\frac{\partial^2 f}{\partial p \partial q} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = 18.
\]

Thus the hessian determinant is equal to \( 36^2 = 18^2 > 0 \) and the first diagonal entry is \( 36 > 0 \). Hence, the hessian matrix is locally positive definite around the critical point, which therefore corresponds to a strict minimum of \( f \). It follows from the

\(^1\text{i.e. an open connected set.}\)
$C^\infty$-equivalent parametrizations, that $\rho$ is minimal only for an equilateral triangle, since $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$. In this case, the radius-ratio is:

$$
\frac{1}{4} f \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \left( \frac{4}{3} \right)^2 \times \frac{3^2}{2} = 2.
$$

3.2. Asymptotic behavior. An obvious link between the radius-ratio and the extremal angles can be deduced from (3.5):

$$
\rho = \frac{\sin \theta_0 + \sin \theta_\infty + \sin (\theta_0 + \theta_\infty)}{2 \sin \theta_0 \sin \theta_\infty \sin (\theta_0 + \theta_\infty)}.
$$

It is therefore interesting to examine how sensitive the radius-ratio is to the information provided by an extremal-angle measurement. In particular, it is well known (cf. [12]) that among all triangulations of the convex hull of a given set of points in $\mathbb{R}^2$, any Delaunay-triangulation maximizes the minimal vertex-angle; this property is generally considered to be a guarantee for the quality of the elements obtained through a Delaunay-based algorithm, while (3.19) shows immediately that this alternative view of triangle-quality is not bijectively linked to the radius-ratio.

Figure 2 shows parts of the surface defined by the dependence of $\rho$ on two angles of the triangle. The radius-ratio is normalized for convenience, since most users tend to prefer that the quality of an equilateral triangle is 1; figure 3 shows the corresponding isovalue.

Figures 2 and 3 clearly indicate a weak dependence between $\rho$ and the extremal angles. Nevertheless, it is possible to derive some results, providing bounds for the radius-ratio measure for given extremal angular values. More precisely, (3.5) gives:

$$
\frac{\partial \rho}{\partial \theta_0} (\theta_0, \theta_\infty) = \frac{-\cos \left( \frac{\theta_0}{2} + \theta_0 \right)}{8 \sin^2 \frac{\theta_0}{2} \sin \frac{\theta_\infty}{2} \cos \frac{\theta_0 + \theta_\infty}{2}}
$$

$$
\frac{\partial \rho}{\partial \theta_\infty} (\theta_0, \theta_\infty) = \frac{-\cos \left( \frac{\theta_\infty}{2} + \theta_\infty \right)}{8 \sin^2 \frac{\theta_0}{2} \sin \frac{\theta_\infty}{2} \cos \frac{\theta_0 + \theta_\infty}{2}}.
$$

It is now straightforward to study the variations of $\rho$ when the minimal angle is fixed: for any given $\theta_0$ in $[0, \frac{\pi}{3}]$, it follows from (3.21) and (2.9) that both the denominator and the numerator of $\frac{\partial \rho}{\partial \theta_\infty}$ are strictly positive for $\theta_\infty$ in the interval $\left[ \frac{\pi}{6} - \frac{\theta_0}{2}, \pi - 2\theta_0 \right]$, since $0 < \theta_0 \leq \frac{\pi}{3}$. Combining with (2.4), it then follows that $\theta_\infty \mapsto \rho(\theta_0, \theta_\infty)$ is a continuous strictly monotonically increasing function over the closed interval $\left[ \frac{\pi}{6} - \frac{\theta_0}{2}, \pi - 2\theta_0 \right]$ and therefore attains a unique minimum (resp. maximum) at the lower (resp. upper) bound of this interval. In other words,

**Proposition 3.1.**

$$
(\forall \theta_0 \in [0, \frac{\pi}{3}]) \quad \inf_{\theta_\infty} \rho = \frac{1 + \sin \frac{\theta_0}{2}}{\sin \theta_0 \cos \frac{\theta_0}{2}}
$$

$$
(\forall \theta_0 \in [0, \frac{\pi}{3}]) \quad \sup_{\theta_\infty} \rho = \frac{1 + \cos \theta_0}{\sin \theta_0 \sin 2\theta_0}.
$$

**Example 3.2.** Let the minimal angle be equal to $10^\circ$ (resp. $20^\circ$), then the normalized radius-ratio ranges from circa 3.14 to 16.7 (resp. 1.74 to 4.41). Hence, even for "reasonable" values of $\theta_0$, it is clear that the radius-ratio is only weakly dependent on the minimal-angle.