A LOCAL MASS CONSERVATION METHOD FOR THE LEVEL SET METHOD APPLIED TO CAPILLARY INCOMPRESSIBLE FLOW

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Abstract. In the case of incompressible flow, the level set method does not ensure mass conservation and it has to be enforced. The usual global mass conservation presents un-accuracy in the case of separated phases of one fluid. In this paper, we propose an alternative local mass conservation based local volume fluxes. We apply our method to test cases where one fluid separates in disconnected parts (motion of two separated liquid volumes, break-up due to surface tension).

1 INTRODUCTION

Interface motion in the case of two-phase incompressible flow simulation is of important interest for applied research. This subject sets a difficult approximation problem: Lagrangian Methods present difficulties to simulate break-up events, Eulerian Methods present difficulties to advect the interface with a good accuracy. In the set of Eulerian Methods, the Level Set Method (LS in the sequel) developed in the early 80’s by Osher and Sethian ([1],[2]) uses the zero level contour of a smooth function to locate the interface opens the possibility to get high accuracy. In case of break-up or coalescence events, the level set method allows reconnection and deconnection of the interface in a natural way with the zero level set contours. However, mass conservation is not naturally satisfied by standard LS. Let us consider the case of an incompressible flow with two immiscible fluids. The volume occupied by each fluid should remain constant. The satisfaction of this
property by an Eulerian method is a rather complex issue. Two different contexts should be kept in mind. When mesh is enough finer than interface smaller detail, conservation is perfectly satisfied by some method like VOF and satisfied up to a small error by non-conservative methods like LS. When an evolution of the interface becomes smaller than mesh size, conservative methods like VOF may produce unphysical artefacts which in some cases are lost for the phase component from which it separated. Non-conservative methods like non-conservative LS may behave even worse, loosing a rather important mass amount. This dilemma has motivated the development of quasi-conservative and conservative LS methods. Globally mass-conserving methods carry some improvement, see for example [11], but, as discussed in the sequel, this kind of improvement does not provide a complete answer to the question risen here.

We also mention shortly the attempts to reduce mass loss in other stage than LS advection, for example the redistancing stage, see [10], but our interest here is in the advection stage.

As remarked in many works, advection can be conservative for the Level Set Function (LS in the sequel) \( \phi \), but this has not directly an application to mass conservation (further, redistancing does not in general conserve \( \phi \)). In [4], a method is proposed for using a parametrisation of interface that will be stiffer than the LS, but still smooth in order to improve the mass conservation at advection step.

In most other approaches, a finite-volume method applied to the characteristic function of a phase is explicitly introduced. In [5], both LS and VOF are combined and produce an hybrid scheme that is a kind of improvement of both methods. The discrete dependant variable for interface is also a synthesis of LS ans VOF representations. The assembly of a VOF method is rather complex and instead, a more simple finite-volume method can be applied to a characteristic field derived from the LS field. This is the idea proposed in [3]. With this kind of approach, the difficult step is to derive a level set from the finite-volume predictor. In [3], this is performed by a Newton pointwise relaxation.

In the work that we propose here, we look for mass conservation in a global algorithm that will apply on unstructured meshes, triangles in 2D with possible extension to tetrahedra in 3D. In that context, the simplicity of a unique, vertex centered approximation is attractive due to its simplicity and ease for second derivatives and then for capillarity modelling. However, the discrete velocity does not satisfy a (discrete or not) zero divergence relation.

For flow with break-up, in particular flow with capillary effect, using a local mass conservation algorithm seems important. First the mass loss is much more important in the break-up occuring area because the accuracy is lower due to discontinuities in the velocity field. Secondly each disconnected phase should keep a constant mass.
In this paper, we present a local mass conservation enforcement method which produces an approximative local conservation. Our method also uses a finite-volume step but only in a very small part of the algorithm as in [3], the dependant variable being essentially the LS function. The method is formulated in the standard P1 finite-element framework in combination with a projection method for Navier-Stokes. It will be applied to liquid oscillations under gravity of two water masses separated by a wall and to the computation with surface tension of a 2D air bubble rising the surface of water.

2 LEVEL SET METHOD FOR TWO-PHASE INCOMPRESSIBLE FLOW

Let us consider the solution of the model of two incompressible immiscible fluids with interface tension for the fluid velocity $\mathbf{U}$, the pressure $p$ and the density $\rho$ :

$$
\rho \frac{\partial \mathbf{U}}{\partial t} + \rho \nabla \cdot (\mathbf{U} \times \mathbf{U}) - \nabla \cdot (2 \nu(\rho) \nabla \mathbf{U}) + \nabla p + \sigma \kappa(\rho) \delta(\rho) \mathbf{n} - \rho g = 0
$$

$$
\partial_t \rho + \nabla \cdot (\rho \mathbf{U}) = 0 \quad \rho = \rho_l \text{ or } \rho_g
$$

$$
\nabla \mathbf{U} = 0
$$

(1)

In this formulation, the density takes only two values in two subdomains separated by an interface smooth enough for allowing to consider its normal:

$$
\mathbf{n} \delta(\rho) = \frac{1}{\rho_l - \rho_g} \nabla \rho,
$$

(2)

further, $\delta(\rho)$ denotes the Dirac delta function on the interface, $\sigma$ the surface tension coefficient, $\kappa(\rho)$ the curvature of the interface, $g$ the gravity volumic force, $\nu(\rho)$ the viscosity and $\rho_l, \rho_g$ the two values taken by the density in each fluid. Typically $\rho_l$ in the liquid and $\rho_g$ in the gas.

The discontinuity in the density is the cause of difficulties in the approximation of the advection of the interface $\Gamma$ separating the two components of different density, in conservation of moments, and in imposing incompressibility. The interface advection can be written with the characteristic function $\chi$ of liquid phase

$$
\frac{\partial \chi}{\partial t} + \nabla \cdot (\mathbf{U} \chi) = 0 \quad (\chi = 0 \text{ or } 1).
$$

(3)

The characteristic function can be used to define the density distribution.

$$
\rho = \chi \rho_l + (1 - \chi) \rho_g.
$$

(4)

The formal accuracy of the advection of a step function as $\chi$ is severely limited to first order unless the numerical scheme cleverly exploits the fact that $\chi$ takes only two different values.
Let $H$ be the step function such that $H(x) = 1$ if $x > 0$ and $H(x) = 0$ elsewhere. The Level Set method introduced by Osher et Sethian ([7]) relies on smoother function $\phi$ such that $\chi = H(\phi)$ with $H(\phi) = \chi\phi$ the Heavyside function on the interface $\Gamma$:

$$\frac{\partial \phi}{\partial t} + V \cdot \nabla \phi = 0 \quad ; \quad \chi = H(\phi) \quad (5)$$

$\phi$ is then described as the signed-distance to the interface $\phi = \pm d(\Gamma)$ ([8],[9]). We take $\phi < 0$ in the gas region and $\phi > 0$ in the liquid region.

The interface is the zero level set of $\phi$.

$$\Gamma = \{ \mathbf{x} \mid \phi(\mathbf{x}, t) = 0 \} \quad (6)$$

We take $\phi < 0$ in the gas region and $\phi > 0$ in the liquid region. Using the $\phi$ function, the previous governing equation for the fluid velocity $\mathbf{U}$ and the pressure $p$ along with boundary conditions can be written as a single equation with a Continuous Surface Force ([6]) formulation of the surface tension term ([10]),

$$\rho(\phi) \frac{D\mathbf{U}}{Dt} = -\nabla p + \nabla \cdot (2\nu(\phi)D) - \sigma \kappa(\phi) \delta(\phi) \nabla(\phi) + \rho(\phi) \mathbf{g} \quad (7)$$

The surface tension force is modelled as a volumic interfacial force on a thickened interface. $\kappa(\phi)$ is the curvature computed in all the domain as the second derivative of $\phi$.

$$\kappa(\phi) = \nabla \cdot (\mathbf{n}) = \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \quad (8)$$

The density and the viscosity are constant in each fluid, we can write

$$\rho(\phi) = \rho_g + (\rho_g - \rho_l) \left[ \frac{H(\phi)}{\rho} \right]$$

$$\nu(\phi) = \nu_g + (\nu_g - \nu_l) \left[ \frac{H(\phi)}{\rho} \right] \quad (9)$$

Let us now to choose a numerical method for the advancing the incompressible velocity. We consider a projection method for the P1-P1 space discretisation:

**Stage 1:** (Prediction) Compute an explicit predictor for moment:

$$\dot{\mathbf{U}} = \mathbf{U}^n - \Delta t \left( \nabla (\mathbf{U} \times \mathbf{U}) - \frac{1}{\rho} \nabla \cdot (2\nu(\phi)\nabla \mathbf{U}) + \frac{1}{\rho} \sigma \kappa(\rho) \delta(\rho) \mathbf{n} - \mathbf{g} \right) \quad (10)$$

**Stage 2:** (Projection) Solve the elliptic system:

$$\nabla \cdot \frac{1}{\rho} \nabla p = -\frac{1}{\Delta t} \nabla \cdot \dot{\mathbf{U}}$$

+ boundary conditions

and put:
\[ U^{n+1} = \bar{U} + \Delta t \mathcal{P} \left( \frac{1}{\rho} \nabla p \right) \]  

(12)

where \( \mathcal{P} \) maps a constant-by-element variable to a \( P_1 \) one.

**Stage 3:** (LS advection) Advect the \( \phi \) function with \( V \) from time level \( n \) to time level \( n + 1 \),

**Stage 4:** (Redistancing) Replace the advected \( \bar{\phi}^{n+1} \) by a reinitialised or redistanced \( \tilde{\phi}^{n+1} \) = signed distance to \( \{ \bar{\phi}^{n+1} = 0 \} \),

**Stage 5:** (Conservation) Replace the re-initialised \( \tilde{\phi}^{n+1} \) by a \( \phi^{n+1} \) enjoying a conservation property.

This deserves some comments:

In stage 1, the vanishing of divergence is re-inforced only for the following approximation of velocity field:

\[ \bar{U} = \mathcal{P} \bar{U} + \frac{1}{\rho} \nabla p \]  

(13)

where \( \mathcal{P} \) is the usual mapping from \( P_1 \) to constant-by-element. Then discrete conservation satisfied by \( \bar{U} \) is set around every vertex \( i \). For an internal vertex, it writes:

\[ \int \bar{U} \cdot \nabla \psi_i = 0 \]  

(14)

where \( \psi_i \) holds for the \( P1 \) basis function at vertex \( i \).

For stage 4, the usual method is a global mass conservation that we recall in section 3. This method is accurate for two-phase incompressible test cases when each fluid is forming an unique volume. In the case where one fluid is disconnected in separated volumes, the global mass conservation does not ensure the conservation of each volume.

### 3 GLOBAL MASS CONSERVATION

An important propriety of incompressible two-phase flow is the volume conservation of each phase which expresses in terms of \( \chi = H(\phi) \):

\[ \chi_t + \nabla.(\chi U) = 0 \Rightarrow \int \chi_t dx = 0 . \]  

(15)

One of the important drawbacks of the level set method is the loss of conservation of this formulation because of the use of \( \phi \). It is possible to conserve \( \phi \) by solving \( \phi_t + \nabla.(\phi U) = 0 \) but this does not imply the conservation of each phase.
In [11], Smolianski proposes a global mass conservation which corrects at each time step $\phi^{n+1}_h$ into $\phi^{n+1}_h$. The idea is to add a small perturbation $C_\phi$ to $\phi^{n+1}$ in each point of the domain to ensure conservation of the constant volume of the liquid phase $V_{liquid} = \int H(\phi^0)d\nu$ with $\phi^0$ being the initial level set.

\[
\phi^{n+1}_h = \phi^{n+1}_h + C_\phi
\]

(16)

\[
\int H(\phi^{n+1}_h)d\nu = V_{liquid} .
\]

(17)

This algorithm uses the property that the level set is a signed distance function ($\nabla \phi = 1$). Adding a constant correction $C_\phi$ will neither change the form of the interface induced by the computed motion, nor change the signed distance function property.

In our version of this algorithm, the $C_\phi$ perturbation is determined by a Newton (regula falsi) method until $\phi^{n+1}_h$ respects exactly the conservation of the volume. As far as $H(\phi_h)$ is an higher order accurate approximation of $\chi$, the error in $\int |H(\phi_h) - \chi|d\nu$ is of higher order and the global volume correction step will not degrade the accuracy.

4 LOCAL MASS CONSERVATION

![Figure 1: Correction of level set function $\phi$ on a triangle $T$](image)

As already mentioned, mass conservation is not satisfied by the LS method under study, due to several reasons:

(i) The discrete velocity field is not divergence free,

(ii) The advection of $\phi$ in general does not satisfy conservation,

(iii) Advection of small details of interface result in truncating them and will not satisfy conservation,

(iv) Redistancing does not in general satisfy conservation.

Point (iii) deserves some comments: moving for example a small isolated bubble around a vertex of one $\Delta x$ results in the disappearing of the bubble, which cannot be
represented by the LS method inside an element. Either the bubble is not moved correctly or it disappears. Our standpoint in this work is not to solve this conflict: we shall let the bubble disappear and compensate the loss globally. We concentrate on an approximate local correction which will compensate only part (ii) of the four previous issues.

4.1 Elementwise analysis

We focus on a particular element $T$. Let us suppose that we start from an interface located with a LS function $\phi^n$.
- the LS method will first advect $\phi$ and produce a $\tilde{\phi}^{n+1}$, in short:
  \[ \frac{1}{\Delta t} (\tilde{\phi}^{n+1} - \phi^n) + \nabla \cdot (U\phi) = 0 \]  
  (18)
- the corresponding characteristic function $\chi^n = H(\phi^n)$ is built and its mean on $T$ is conservatively advanced in time as follows:
  \[ \frac{1}{\Delta t} \int_T (\chi^{n+1} - \chi^n) \, dx = - \int_{\partial T} \chi U \cdot n \, ds. \]  
  (19)

We observe (P1 exactness) that with and adequate integration of $\phi$ (typically $\phi = \tilde{\phi}^{n+1/2}$), this step can be exact for a straight interface and a uniform velocity. We look for a LS function $\phi_T$ such that:
  \[ \int_T H(\phi_T) \, dx = \int_T \chi^{n+1} \, dx \]  
  (20)

It is enough to put:
  \[ \phi_T = \tilde{\phi}^{n+1} + \frac{1}{||\nabla \phi^{n+1}||} \, h \]  
  (21)

where $h$ is such that the surface of trapezoid between interfaces for $\phi^{n+1}$ and for $\phi_T$ is exactly the conservation error (Fig.1):
  \[ \int_T (H(\phi_T) - \chi^{n+1}) \, dx = 0. \]  
  (22)

Remark 1: Consistency by P1 exactness: we observe that in the case where $\chi^{n+1}$ corresponds to the uniform advection of $H(\tilde{\phi}^{n+1})$, the correction will be exact, that is we get $H(\phi_T) = \chi^{n+1}$.

4.2 Correction on all vertices

In any element in which the previous study could be done, we shall get some candidate for the new LS $\phi^{n+1}$ or equivalently for correcting the LS $\tilde{\phi}^{n+1}$. This means that corrections are available only on vertices belonging to the triangles in which the element-wise
analysis of previous subsection gives a reasonable correction.

**Remark 2:** in the case of Remark 1, all the corrections are equal and P1 exactness holds globally.

In the general case, we choose to compute a mean between the different available corrections for a same vertex value. This produces a vertex-wise correction that is directly applied to \( \bar{\phi}_{i}^{n+1} \) in order to reduce the local error of conservation.

\[
\bar{\phi}_{i}^{n+1} = \bar{\phi}_{i}^{n+1} + \frac{1}{||\nabla \bar{\phi}_{i}^{n+1}||} \ h_{i} \ , \ \forall \ i, \text{vertex} \ .
\]  

(23)

As already mention, this procedure is not able to correct all conservation errors. The local correction step is then followed by a redistancing step (not at every time step) and a global conservation correction.

5 NUMERICAL EXPERIMENTS

5.1 Liquid oscillation in two separated tanks

As a first test case to compare both mass conservation methods, we compute the sloshing of a mass of water under gravity \( (g = 9.8 \ m/s^2) \). The 1 meter large tank is made of two parts separated by a thick wall. The fluid are assumed to be inviscid and the density ratio is 1 : 1000. The initial shape of air-water interfaces are assumed to be a stable horizontal plan in the right sub-tank and a sine curve (cf. Fig.2). As there is a motion in the left tank and none in the right one, all the errors on the mass are done in the left. The total number of vertices is 1250 for coarse mesh and 5000 for the fine one. By construction, the global mass conservation algorithm corrects the mass lost in right side by sharing it uniformly between both side. Fig.3 shows the interface position at \( t = 1 \ s \) for both methods on the two meshes. When applying the local correction, the level on right stays at the exact level. For the fine mesh the global correction (in green) has spuriously affected the level on right, and the spurious deviation is about twice larger for the coarse calculation. Fig.4 depicts the evolution in time of the volume loss in the right tank for the two options, global mass conservation method and local mass conservation on the two embedded meshes. Errors for local conservation are about 1\% for the coarse mesh used, in contrast to the 4 – 8\% produced by the global conservation.

5.2 Air bubble rising on surface

While the evolution of interfaces without capillary tension does not tend to produce break-ups, on the contrary, capillary forces will. After break up, small phase components are either immediately re-attached to the main component, or transformed into a quasi-spherical bubble/drop and start a travel under the influence of ambient flow and gravity.
Figure 2: Initialisation of the “two separated tanks” test case

Figure 3: Interface location at $t = 1s$
We try to see how our algorithm can avoid two spurious events:
- the distribution of lost mass on the whole interface produced by the global conservation,
- the creation of too small components that will not move in a consistent manner, as happen in case of some strictly conservative methods.

We consider the rising of a 2D air bubble up to an horizontal interface. The physical consitions are the same as the previous calculation, except that the interface tension coefficient is \( \sigma = 0.072 \) SI, and the initial diameter of the bubble is \( 5 \times 10^{-3} m \), and so is the distance from its center to the interface (Fig.5).

For the global conservative calculation, the effect of spurious mass loss on the shape of the bubble near the surface is rather striking for a coarse mesh of 2500 vertices (Fig.6). This is much improved in the twice finer calculation with 10,000 vertices (Fig.7). The coarse-mesh locally conservative calculation shows good trends and in particular the good shape of the bubble till breaking (Fig.8).

6 CONCLUSIONS

In this paper, we have presented a local mass conservation algorithm for a Level Set Method applying on unstructured meshes. The method is based on a correction of the error in mass fluxes on mesh elements.

Comparisons with a non-local conservative option are performed for the sloshing of a liquid in a two-part tank. We observe important improvements, in particular for the conservation of volumes of separate components of each phase. Comparisons are also
Figure 5: Rising bubble: initial condition.

Figure 6: Rising bubble: interface location at different times around merging time with the global mass conservation on coarse mesh.
Figure 7: Rising bubble: interface location at different times around merging time with the global mass conservation on fine mesh

Figure 8: Rising bubble: interface location at different times around merging time with the local mass conservation on coarse mesh
performed for a flow with capillary effect, the rising of a bubble under surface tension. The overall shapes for the new method are also closer to those obtained with fine-mesh results. The method extends rather easily to 3D. Current and future work involve 3D calculations and mesh adaptive ones.

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