On Some Geometric Inverse Problems in Linear Elasticity

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Abstract

The aim of this paper is to derive identifiability results and stability estimates for inverse problems in linear elasticity, namely the detection of inclusions or interfaces. We consider to different cases in two spatial dimensions, obtained by reduction of the three-dimensional case in a planar and anti-planar case.

We prove unique identifiability of inclusions and interfaces from a boundary measurement of the displacement in a rather general setting. Moreover, we derive directional Lipschitz stability estimates for transformed shapes, using shape and material derivatives for the least-squares problems associated to these inverse problems.

Keywords: Geometric Inverse Problems, Linear Elasticity, Identifiability, Stability

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1 Introduction

This paper is devoted to the mathematical analysis of some geometric inverse problems in linear elasticity, namely the identification of interfaces or inclusions (with possibly multiple connected components) from boundary measurements of the displacement. We develop a rather general approach for identifiability and local Lipschitz stability for such geometric inverse problems, where the normal component of the stress tensor satisfies a homogeneous boundary condition on the unknown geometry. The boundary measurements in such applications are those of displacement on a part of the boundary where we have a boundary condition satisfied by the normal component of the stress tensor, too. We shall consider two

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two-dimensional cases derived from the three-dimensional problems for specific geometrical situations and special applied loads, the so-called \emph{planar} and \emph{anti-planar} cases.

Uniqueness and stability for geometric inverse problems, i.e., problems where the unknown is a geometric shape, are a classical subject of theoretical studies (cf. e.g. Dervieux and Palmerio [10], Kohn and Vogelius [17]), for results on problems close to the one considered here, we refer to Alessandrini et al. [2, 3, 4] and Hettlich, Rundell [13]. Whereas most of the existing literature on stability results for these problems is focused on global estimates for simply connected shapes (usually of logarithmic type), we take a different approach deriving directional Lipschitz stability for multiply connected shapes. Similar directional stability estimates have been obtained recently for crack detection problems by Ben Abda, Ben Ameur, and Jaoua [5].

Besides practical applications, a motivation for this approach is the recent development of numerical methods being able to solve inverse obstacle problems with multiply connected obstacles (cf. e.g. Kirsch [16], Santosa [22]). In particular our approach corresponds very well to the spirit of the \emph{level set method} (cf. Osher and Fedkiw [20] for a general overview, and Burger [7], Santosa [22] for inverse obstacle problems), where the shape is evolved toward a solution by a geometric motion, with appropriately chosen normal velocity. We refer to Ben Ameur, Burger and Hackl [6] for the numerical solution of the geometric inverse problems considered here by the level set methods. In regular situations, the directional stability results obtained in this paper can be applied to obtain stability of the level set method.

The remainder of the paper is organized as follows: In Section 2, we give a precise formulation of the direct problems and their weak formulations as well as the inverse problems. In Section 3, we prove identifiability results for multiply connected shapes, based on the Almansa lemma. Using similar technique together with shape derivatives, we are able to obtain local directional stability estimates in Section 4.

2 \hspace{1em} Direct problem and Inverse Problem

In the following we consider a homogeneous isotropic linear elastic material, in a domain $\mathcal{D} \subset \mathbb{R}^3$, assuming that there exists a surface $\Gamma \subset \mathcal{D}$ that separates the domain into two disjoint open sets $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e.,

$$\overline{\mathcal{D}} = \overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2}, \quad \Gamma = \partial \mathcal{D}_1 \setminus \partial \mathcal{D} = \partial \mathcal{D}_2 \setminus \partial \mathcal{D}.$$  

The linear elasticity problem under consideration is specified by

\begin{align}
\text{div} \sigma(u) &= 0 & \text{in } \mathcal{D} \setminus \Gamma = \mathcal{D}_1 \cup \mathcal{D}_2 \\
\sigma(u) \cdot n &= 0 & \text{on } \Gamma
\end{align}  

(2.1)

supplemented by appropriate boundary conditions (specified below) on $\partial \Omega$, $n$ denotes a normal vector to $\Gamma$. In the following we will use the convention that this normal vector is in the outward normal direction of $\mathcal{D}_1$. The vector $u$ denotes the displacement and $\sigma(u)$ is the associated stress tensor, which is related via Hooke's law to the linearized strain tensor $\epsilon(u)$ via

$$\sigma = \lambda \text{tr} \epsilon I + 2\mu \epsilon.$$  

(2.2)
The linearized strain tensor $\epsilon(u)$ is given by

$$\epsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

tr denotes the trace of a matrix, and $\lambda, \mu$ are Lamé coefficients related to Young’s modulus $E$ and the Poisson ratio $\nu$.

$$\mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E \nu}{(1-2\nu)(1+\nu)}.$$

The identification process consists in applying some prescribed load $g$ on $\Gamma_N \subset \partial \Omega$ and measuring the displacement induced by $g$ on some part $\Gamma_M$ of $\Gamma_N$. $\Gamma_M$ having a strictly positive measure.

### 2.1 Planar and Anti-Planar Cases

We assume that $\Omega = \Omega \times \mathbb{R}$, where $\Omega$ is a bounded domain in $\mathbb{R}^2$, $\Omega = \Omega_1 \cup \Omega_2$ and $\Sigma = \partial \Omega_1 \cap \partial \Omega_2$. We suppose that we apply a planar load $g$ and that the displacement $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^3$ depends only on $x_1$ and $x_2$. We split the initial 3D problem (2.1) into two problems: A planar strain one, where we consider $u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$, and an anti-planar problem depending only on the third component $u(x_1, x_2) = (0, 0, u_3(x_1, x_2))$. In the latter case the original linear elasticity problem (2.1) is nothing else than the Laplace problem:

$$\begin{align*}
\mu \Delta u_3 &= 0 & \text{in } \Omega \setminus \Sigma \\
\mu \frac{\partial u_3}{\partial n} &= g & \text{on } \Gamma_N \\
\mu \frac{\partial u_3}{\partial n} &= 0 & \text{on } \Sigma \\
u_3 &= 0 & \text{on } \Gamma_D
\end{align*}$$

(2.3)

where $\mu$ is Lamé constant. From now on, for simplicity, we denote $u_3$ by $u$ in (2.3).

In the planar strain case, we obtain a stress tensor $\sigma$ satisfying $\sigma_{3,l} = \sigma_{l,3} = 0, l = 1, 2$. The choice of $g$ with a third component equal to zero will allow us to obtain a two-dimensional problem similar to the one corresponding to the plane stress case where $\sigma_{3,l} = \sigma_{l,3} = 0, l = 1, 2, 3$. We call both of these two cases “planar case”. Hence, in the planar case (2.1) can be written:

$$\begin{align*}
\text{div } \sigma &= 0 & \text{in } \Omega \setminus \Sigma \\
\sigma \cdot n &= g & \text{on } \Gamma_N \\
\sigma \cdot n &= 0 & \text{on } \Sigma \\
u &= 0 & \text{on } \Gamma_D
\end{align*}$$

(2.4)

where $\{\Gamma_D, \Gamma_N\}$ is a partition of the boundary of $\Omega$ supporting respectively Dirichlet and Neumann boundary conditions.

### 2.2 Weak Formulation of the Direct Problem

In the following we introduce weak formulations of the direct problems in a classical Sobolev space framework, which is appropriate for our tasks since we shall assume that the connected components of $\Sigma$ are smooth (piecewise $C^1$). Note that for more general (Hausdorff-measurable) interfaces $\Sigma$ one has to use a framework in Deny-Lions spaces (cf. e.g. Chambolle [8]).
2 DIRECT PROBLEM AND INVERSE PROBLEM

![Figure 1: Different situations]

We start with the weak formulation of the anti-planar problem in Sobolev spaces. For this sake we assume that $\Sigma$ is sufficiently regular and introduce the subspace

$$\mathcal{U} := \{ u \in H^1(\Omega \setminus \Sigma) \mid u|_{\Gamma_D} = 0 \}.$$ 

The weak formulation of the anti-planar case consists in finding $u \in \mathcal{U}$ such that

$$a(u, v) := \int_{\Omega \setminus \Sigma} \nabla u \cdot \nabla v \, dx = \int_{\Gamma_N} g v \, ds =: \langle G, v \rangle \quad \forall u \in \mathcal{U}.$$ 

The solution of this problem is also the unique minimizer of the variational problem

$$a(u, u) - 2\langle G, u \rangle \to \min_{u \in \mathcal{U}}.$$ 

A similar weak formulation is obtained in the anti-planar case with the space

$$\mathcal{U} := \{ u \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2) \mid u|_{\Gamma_D} = 0 \}$$

and the variational equation

$$a(u, v) := \int_{\Omega \setminus \Sigma} \sigma(u) : \epsilon(v) \, dx = \int_{\Gamma_N} g v \, ds =: \langle G, v \rangle \quad \forall u \in \mathcal{U},$$

with $\sigma$ and $\epsilon$ defined as above and $\sigma : \epsilon = \sum_{i,j} \sigma_{ij} \epsilon_{ij}$.

### 2.3 Inverse Problem

In the following we shall assume that all interfaces satisfy

$$\Sigma = \bigcup_{j=1}^{J} (\Sigma_j \cap \Omega), \quad (2.5)$$

for some positive integer $J$, where $\Sigma_j$ is the boundary of a $C^1$ embedded surface, and the union is disjoint. As a direct consequence, the connected components of the interface are either the boundary of a subset of $\Omega$ (inclusions) or curves separating the domain $\Omega$ into two subdomains (interfaces), see Figure 1.
Under this rather general geometric assumption, we consider the following inverse problems:

**Inverse Problem, Anti-Planar Case:** Identify the unknown interface $\Sigma$ from a measurement of the vertical displacement $u$ on $\Gamma_M \subset \Gamma_N$ (with $\Gamma_M$ having a strictly positive measure), where $u$ is the solution of (2.3).

**Inverse Problem, Planar Case:** Identify the unknown interface $\Sigma$ from a measurement of the displacement $u$ on $\Gamma_M \subset \Gamma_N$ (with $\Gamma_M$ having a strictly positive measure), where $u = (u_1, u_2)$ is the solution of (2.4) with the constitutive law (2.2).

3 Identifiability

Many theoretical studies have been performed for problems close to ours. Kohn and Vogelius [17] established uniqueness, with infinitely many measurements, for inclusion domains with analytical boundaries. Later Isakov [14] provided the same result for Lipschitz boundaries. Alessandrini [1], Isakov and Powell [15] and Kirsch [16] studied uniqueness result for a finite number of measurements with different assumption on the set of admissible geometries.

3.1 The Anti-Planar Case

As in Kirsch [16] and Hettlich and Rundell [12], the main tool for proving an identifiability result is Holmgren’s theorem. By an application of Holmgren’s theorem we can easily prove the following identifiability result:

**Theorem 3.1 (Identifiability in the anti-planar case).** Let $\Sigma$ and $\tilde{\Sigma}$ be two interfaces in $\Omega$, and let $u$, $\tilde{u}$ be the corresponding solutions of the direct problem (2.3), in $\Omega$ and $\tilde{\Omega}$ with load $g$ ($g \neq 0$), where $\Omega = \Omega_1 \cup \Omega_2$, $\Sigma = \partial \Omega_1 \cap \partial \Omega_2$ and $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$, $\tilde{\Sigma} = \partial \tilde{\Omega}_1 \cap \partial \tilde{\Omega}_2$. Then, if both $\Sigma$ and $\tilde{\Sigma}$ result into the same measurements on $\Gamma_M$, we have $\Sigma = \tilde{\Sigma}$.

Since most parts of the proof are analogous or even simpler than in the planar case, we shall sketch it after the proof of Theorem 3.3.

3.2 The Planar Case

Holmgren’s theorem is not valid in the planar case, where we have to deal with a system of elliptic equations. However, there is a similar result for the elasticity system known as the Almanzi lemma (Muskhelishvili [19]):

**Lemma 3.2 (Almanzi).** Let $D$ be a domain in $\mathbb{R}^3$ and $(\sigma, u)$ be the solution of the linear elastic problem (2.1), (2.2). Assume that there exists some $C^{1,1}$ open subset $L$ of $\partial D$ such that:

$$\sigma \cdot n = 0 \quad \text{on } L$$

$$u = 0 \quad \text{on } L$$

Then, $\sigma \equiv 0$ and $u \equiv 0$ in $D.$
We want to identify the interface $\Sigma$ in different situations (Figure 2) using the two dimensional linear elastic problem (2.4)

**Theorem 3.3 (Identifiability).** Let $\Sigma$ and $\tilde{\Sigma}$ be two interfaces in $\Omega$ satisfying assumption (2.5), and let $u$, $\tilde{u}$ be the corresponding solutions of the direct problem (2.2), (2.4) in $\Omega$ and $\tilde{\Omega}$ with load $g$ ($g \neq 0$), where $\Omega = \Omega_1 \cup \Omega_2$, $\Sigma = \partial \Omega_1 \cap \partial \Omega_2$ and $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$, $\tilde{\Sigma} = \partial \tilde{\Omega}_1 \cap \partial \tilde{\Omega}_2$. Then, if both $\Sigma$ and $\tilde{\Sigma}$ result into to the same measured displacement on $\Gamma_M$, we have $\Sigma = \tilde{\Sigma}$.

**Proof.** Let us assume without restriction of generality that $\text{meas}(\Gamma_M \cap \partial \Omega_1)$ is not equal to zero. Consider $\tau = \sigma - \tilde{\sigma}$ and $w = u - \tilde{u}$. Then:

$$
\begin{align*}
\text{div } \tau &= 0 & \text{in } \Omega_1 \cap \tilde{\Omega}_1 \\
\tau \cdot n &= 0 & \text{on } \Gamma_M \cap \partial \Omega_1 \\
w &= 0 & \text{on } \Gamma_M \cap \partial \Omega_1 \\
\end{align*}
$$

(3.1)

where $\tau$ and $w$ are related by Hooke's law. From Almansi's lemma (3.2) we can deduce that:

$$
\begin{align*}
\sigma &= \tilde{\sigma} & \text{in } \Omega_1 \cap \tilde{\Omega}_1 \\
u &= \tilde{u} & \text{in } \Omega_1 \cap \Omega_1 \\
\end{align*}
$$

Now let $\mathcal{O}$ be a connected component of $\left(\Omega_1 \cup \tilde{\Omega}_1\right) \setminus \left(\Omega_1 \cap \tilde{\Omega}_1\right)$. In any way $\partial \mathcal{O}$ contains a part of $\Sigma$ and a part of $\tilde{\Sigma}$, so that we have in $\mathcal{O}$:

$$
\begin{align*}
\text{div } \sigma &= 0 & \text{in } \mathcal{O} \\
\sigma &= C(\varepsilon(u)) & \text{in } \mathcal{O} \\
\sigma.n &= 0 & \text{on } \partial \mathcal{O} \cap \Sigma \\
\end{align*}
$$

Due to the continuity of $\sigma.n$ across $\tilde{\Sigma}$ and $\sigma.n = \tau.n + \tilde{\sigma}.n = 0$ on $\partial \mathcal{O} \cap \tilde{\Sigma}$, we deduce that $\sigma.n = 0$ on $\partial \mathcal{O}$. Hence $u$ is a rigid displacement in $\mathcal{O}$. Let $U$ be the extension of this rigid displacement to the whole of $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$, then we obtain that:

$$
\begin{align*}
u - U &= 0 & \text{on } \Sigma \cap \partial \mathcal{O} \\
\sigma(u - U).n &= 0 & \text{on } \tilde{\Sigma} \cap \partial \mathcal{O} \\
\end{align*}
$$

By the Almansi lemma 3.2, we derive $u - U \equiv 0$ in $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$, and therefore $\sigma.n = 0$ on $\Gamma_N$. This is in contradiction to the assumption that the load $g$ is not identically to zero ($g \neq 0$). \qed
Remark 1. We note that in the anti-planar case (3.1) becomes:

\[
\begin{align*}
\mu \Delta w_3 &= 0 \quad \text{in } \Omega_1 \cap \tilde{\Omega}_1 \\
\frac{\partial w_3}{\partial n} &= 0 \quad \text{on } \Sigma \\
\frac{\partial w_3}{\partial n} &= 0 \quad \text{on } \Gamma_M \cap \partial \Omega_1 \\
w_3 &= 0 \quad \text{on } \Gamma_M \cap \partial \Omega_1
\end{align*}
\]

where \( w_3 \) is the third component of \( w \). Then, by Holmgren’s theorem we deduce that \( w_3 = 0 \)
in \( \Omega_1 \cap \tilde{\Omega}_1 \), and so \( w = 0 \) in \( \Omega_1 \cap \tilde{\Omega}_1 \). Now we can proceed in the same way as in the above proof to deduce the identifiability result of Theorem 3.1.

4 Stability Results

In the following we prove local, directional stability estimates for the inverse problems in the planar and anti-planar case under rather general assumptions on the topology. In particular, we do not restrict the number of connected components of the inclusions, which we assume to be an arbitrary positive integer.

4.1 Shape Derivatives

Like the Gateaux- and Fréchet derivative in a functional analytic framework, the shape derivative is a fundamental tool for geometric inverse problems, since it allows to characterize extrema and yields directions of steepest descent. For a comprehensive introduction to Shape derivatives we refer to Sokolowski and Zolésio [23] and Delfour and Zolésio [9]. The intention of this section is to give a very brief introduction into shape derivatives and provide the later needed shape derivatives for the planar and anti-planar case, that play a crucial role in the stability proofs.

In the following we suppose without restriction of generality that \( \text{meas}(\Gamma_N \cap \partial \Omega_1) > 0 \) and we assume that \( \Sigma \) is piecewise \( C^2 \). Let us define a family of diffeomorphisms

\[
F_t : \Omega \to \Omega_t \subset \mathbb{R}^2 \quad F_t = I + tV,
\]

where \( t \in \mathbb{R}^+ \) is small, \( t \leq t_0 \) and \( V \in C^1 \). This parameterization of domains was first studied by Murat and Simon [18] and Pironneau [21] and enables us to view geometric objects in the usual functional analytic context.

Let \( v_t \) be a function defined on the domain \( \Omega_t \) (e.g. \( v_t = (\sigma_t, u_t) = (\sigma, u)(\Omega_t) \), where \((\sigma, u)(\Omega_t)\) is the solution to (2.3) or (2.4) with \( \Omega \) replaced by \( \Omega_t \)). Then we can consider

\[
\begin{align*}
v' &= \frac{\partial v_t}{\partial t}|_{t=0} \quad \text{shape derivative} \\
v^* &= \frac{\partial v_t}{\partial t}|_{t=0} := \frac{\partial v_{t=0} F_t}{\partial t}|_{t=0} \quad \text{material derivative}.
\end{align*}
\]

It is well known that there is a relation between the material derivative \( v^* \) and the shape derivative \( v' \) (see Destuynder and Jaoua [11] or Sokolowski and Zolésio [23]), namely:

\[
v' = v^* - \nabla v.V.
\]

Before we start to calculate the shape derivative for our problem we note that in our problem class of inclusion and interface identification we want that the boundary \( \partial \Omega \) remains unchanged within the family of diffeomorphisms \( F_t \). Hence, we may restrict our attention
to velocity fields satisfying \( V \cdot n = 0 \) on \( \partial \Omega \setminus \Sigma \), where \( n \) denotes the normal vector to the boundary \( \partial \Omega \setminus \Sigma \). Furthermore it suffices to consider only diffeomorphisms \( F_i \) that change the position of \( \Sigma \) but do not “rotate” \( \Sigma \), which results into vector fields \( V \) that have zero tangential part along \( \Sigma \), i.e.

\[
V|_{\Sigma} = (V \cdot n)n = V_N n.
\]

### 4.2 The Anti-Planar Case

The main tool in the stability proof is the following characterizatoin of the shape derivative \( u' \) given by Sokolowski and Zolésio [23]:

**Theorem 4.1.** Let \( u \) be the solution to the anti-planar case (2.3), \( V_N \in C^1(\Sigma) \) then the shape derivative \( u' \) is determined by the unique solution of:

\[
\begin{align*}
\Delta u' &= 0 & \text{in } \Omega \setminus \Sigma \\
\frac{\partial u'}{\partial \nu} &= 0 & \text{on } \Gamma_N \\
\frac{\partial u'}{\partial \nu} &= -\text{div}_V(V_N \nabla_T u) & \text{on } \Sigma \\
u' &= \frac{\partial u'}{\partial \nu} V_N & \text{on } \Gamma_D
\end{align*}
\]  

(4.2)

where \( \text{div}_T \) and \( \nabla_T \) are defined via: Let \( U \) be an open neighbourhood of \( \Sigma \) and \( Y \in C^1(U, \mathbb{R}^n) \), then

\[
\begin{align*}
\text{div}_T Y &= \text{div}Y|_{\Sigma} - (DY,n)|_{\Sigma} \\
\nabla_T Y &= \nabla Y|_{\Sigma} - \frac{\partial Y}{\partial n}|_{\Sigma} n
\end{align*}
\]

When we multiply (4.2) by some testfunction \( v \in H^2(\Omega) \) and do some manipulations, it turns out that the shape derivative of \( u' \) fulfills:

\[
\int_{\Omega} \nabla u' \cdot \nabla v \ dx = \int_{\Sigma} \nabla_T u \cdot \nabla_T v \ V_N \ ds
\]

(4.3)

The local stability result is then :

**Theorem 4.2 (Local stability in the anti-planar case).** Let \( u \) and \( u_t \) be the solutions of (2.3) in \( \Omega \) and \( \Omega_t \), respectively, and let \( u^t = u_t \circ F_i \) for some diffeomorphisms \( F_i = I + tV \) mapping \( \Omega \) onto \( \Omega_t \), \( V \in C^1 \). If we denote by \( u|_{\Gamma_M}^t \) and \( u|_{\Gamma_M}^t \), respectively, the traces of \( u \) and \( u^t \) on \( M \), then

\[
\lim_{t \to 0} \frac{\|u^t - u\|_{L^2(\Gamma_M)}}{|t|} > 0.
\]

(4.4)

\[\text{Proof.}\] Without loss of generality we restrict ourselves to the case \( \text{meas}(\Sigma \cap \{ Vn \geq 0 \}) > 0 \). By the definition of the material derivative the limes in equation (4.4) is given by:

\[
\lim_{t \to 0} \frac{\|u^t - u\|_{L^2(\Gamma_M)}}{|t|} = \|u^*\|_{L^2(\Gamma_M)} = \frac{\|u^* + \nabla u \cdot V\|_{L^2(\Gamma_M)}}{\|u^*\|_{L^2(\Gamma_M)}}
\]

Due to the choice of \( V \) we know that \( V|_{\Gamma_M} = 0 \). Hence it is enough to proof that \( \|u'\|_{L^2(\Gamma_M)} > 0 \). Assume that \( \|u'\|_{L^2(\Gamma_M)} = 0 \), then the characterization of the shape derivative \( u' \) (4.2)

\[
\begin{align*}
\Delta u' &= 0 & \text{in } \Omega \setminus \Sigma \\
\frac{\partial u'}{\partial \nu} &= 0 & \text{on } \Gamma_M \\
u' &= 0 & \text{on } \Gamma_M
\end{align*}
\]

\[\text{implying that } u' = 0 \text{ on } \Omega \setminus \Sigma \text{ and } u' = 0 \text{ on } \Gamma_M. \]**
and with Holmgren’s theorem we get that $u' = 0$ in $\Omega$. Consequently equation (4.3) reduces to
\[
\int_{\Sigma} (\nabla u \cdot \nabla v) V.n = 0 \quad \forall v \in C^\infty_{0,D}(\Omega)
\] (4.5)
where we additionally used that $\frac{\partial \eta}{\partial n} = 0$. When we can prove that $\nabla u = 0$ on some open subset of $\Sigma \cap \{Vn > 0\}$ we get again from Holmgren’s theorem that $u = 0$ in all of $\Omega$ which would be a contradiction to $g \neq 0$ on $\Gamma_N$. To do so we chose in (4.5) a specific sequence of testfunctions $v$, namely:

Let $c > 0, \delta > 0$, and $\chi_\delta \in C^2(\bar{\Omega})$ satisfy:
\[
\chi_\delta = \begin{cases} 
1 & \text{if } V.n \geq \delta \\
0 & \text{if } V.n \leq 0 
\end{cases} \quad \text{and} \quad 0 \leq |\nabla(\chi_\delta)| \leq \frac{c}{\delta}
\]

Take $v = \chi_\delta u$, then we have
\[
0 = \int_{\Sigma} \nabla u \cdot \nabla(\chi_\delta u) V.n
\]
\[
\Leftrightarrow 0 = \int_{\Sigma \cap \{Vn \geq \delta\}} \nabla u \cdot \nabla(\chi_\delta u) V.n + \int_{\Sigma \cap \{0 \leq Vn \leq \delta\}} \nabla u \cdot \nabla(\chi_\delta u) V.n
\]
\[
\Rightarrow \int_{\Sigma \cap \{Vn \geq \delta\}} (\nabla u)^2 V.n = \left| \int_{\Sigma \cap \{0 \leq Vn \leq \delta\}} \nabla u \cdot \nabla(\chi_\delta u) V.n \right| \leq C \text{meas}(\Sigma \cap \{0 < Vn \leq \delta\})
\]

Due to the choice $\text{meas}(\Sigma \cap \{Vn \geq 0\}) > 0$ and the fact that the term $\text{meas}(\Sigma \cap \{0 < Vn \leq \delta\})$ tends to 0 with $\delta \to 0$ we conclude that $\nabla u = 0$ on $\Sigma \cap \{Vn > 0\}$ and we are done. \[\Box\]

### 4.3 The Planar Case

In the same way as proven by Ben Abda, Ben Ameur and Jaoua [5] the shape derivative $(u', \sigma')$ gets:

**Theorem 4.3.** Let $(u, \sigma)$ be the solution to the planar case (2.4), $V_N \in C^1(\Sigma)$ then the shape derivative $(u', \sigma')$ is determined by the unique solution of:
\[
\begin{align*}
\text{div} \sigma' &= 0 & \text{in } \Omega \setminus \Sigma \\
\sigma' \cdot n &= 0 & \text{on } \Gamma_N \\
\sigma' \cdot n &= -\text{div}_T (V.n \sigma_T) & \text{on } \Sigma \\
u' &= 0 & \text{on } \Gamma_D
\end{align*}
\] (4.6)

where $\sigma_T = \sigma n - (n^t \sigma n)n$.

When we again multiply (4.2) by some testfunction $v \in [H^2(\Omega)]^2$, it turns out that the shape derivative of $u'$ fulfills:
\[
\int_{\Omega} \sigma' \cdot \nabla v = \int_{\Sigma} \sigma_T : \nabla_T v V_N \, ds
\] (4.7)

The local stability result is then the following:
Theorem 4.4 (Stability in the planar case). Let \((u, \sigma)\) and \((u_t, \sigma_t)\) be the solutions of (2.2), (2.4) respectively in \(\Omega\) and \(\Omega_t\) and let \(u^t = u_t \circ F_t\) for some diffeomorphisms \(F_t = I + tV\) mapping \(\Omega\) on \(\Omega_t\), \(V \in C^1\). If we denote by \(u|_{\Gamma_M}\) and \(u^t|_{\Gamma_M}\) respectively the restrictions of \(u\) and \(u^t\) to \(\Gamma_M\), then we have the stability result:

\[
\lim_{t \to 0} \frac{\|u^t - u\|_{L^2(\Gamma_M)}}{|t|} > 0 \tag{4.8}
\]

Proof. As in the anti-planar case we deal, without loss of generality, with the case \(\text{meas}(\Sigma \cap \{Vn \geq 0\}) > 0\). Again the definition of the material derivative gives that

\[
\lim_{t \to 0} \frac{\|u^t - u\|_{L^2(\Gamma_M)}}{|t|} = \|u^*\|_{L^2(\Gamma_M)} \tag{4.1} \leq \|u^t + \nabla u \cdot V\|_{L^2(\Gamma_M)}
\]

The restriction to velocities with \(V|_{\Gamma_N} = 0\) reduces our conjecture to \(\|u^t\|_{L^2(\Gamma_M)} > 0\). When we assume that this conjecture is wrong we get as above from Almansi’s lemma 3.2 \((u^t, \sigma^t) = 0\) in \(\Omega\) and from equation (4.7) and \(\sigma^t \cdot n = 0\) on \(\Sigma\)

\[
\int_{\Sigma} \sigma : \nabla v \cdot V \cdot n \, ds = 0 \quad \forall \, v \in [C_0^\infty(\Omega)]^2. \tag{4.9}
\]

Again we are lead to construct a sequence of testfunctions \(v\), which can be chosen like in the ”anti-planar” case, such that we can conclude that \(\sigma = 0\) on an open set of \(\Sigma \cap \{Vn > 0\}\). Together with the Almansi lemma 3.2 we get that \((u, \sigma) = 0\) in \(\Omega\) which is in contradiction to \(g \neq 0\). \(\square\)

Remark 2. Theorems 4.2 and 4.4 prove the “local continuity” of the map:

\[
\mathcal{F} : L^2(\Gamma_M) \longrightarrow \Sigma_{ad}
\]

\[
u|_{\Gamma_M} \longmapsto \Sigma_t
\]

\((0 \leq t \leq t_0)\) and \(\Sigma_{ad}\) is the set of admissible geometries equipped with an appropriate topology. Let us consider the topology induced by the Hausdorff distance \(d_H\) (cf. Delfour and Zolésio [9] for detailed definition). We have

\[
d_H(\Sigma_t, \Sigma) \leq C|t|, \quad C = \sup_{\{x \in \Sigma\}} |V(x)| \tag{4.10}
\]

and thus, from Theorems 4.2 or 4.4 and inequality (4.10) we deduce “local directional continuity” of \(\mathcal{F}\), since for any admissible \(V\), there exists a constant \(c(V)\) such that

\[
d_H(\Sigma_t, \Sigma) \leq c(V)\|u_t - u\|_{L^2(\Gamma_M)}
\]

for all \(t\) sufficiently close to zero.

Local stability estimates can be obtained under additional assumptions on the regularity of \(V\), which allows to use compactness arguments. Since we are comparing shapes \(\Sigma\) of class \(C^2\), it seems natural to consider variations \(V \in C^2\), because for less smooth variations one cannot guarantee that \(\Sigma_t\) is of class \(C^2\) anyway:
Theorem 4.5. Let $V$ be an arbitrary element of the set $\mathcal{B}_1 := \{ V \in C^2(\Sigma) \mid \| V \|_{C^2} \leq 1 \}$ and let the assumptions of Theorem 4.2 or 4.4, respectively, be satisfied. Then, for some constant $c_0$ (independent of $V$) and for all $t \leq t_0$ (for some $t_0$ sufficiently small), the estimate

$$d_H(\Sigma_t, \Sigma) \leq c_0\| u_t - u \|_{L^2(\Gamma_M)}, \quad \forall V \in \mathcal{B}_1,$$

(4.11)

holds.

Proof. Above we have seen that (4.11) holds for all $V$ of class $C^1$ with a constant $c(V)$ dependent on $V$. Now suppose that there exists a sequence $V_k \in \mathcal{B}_1$ such that $c(V_k) \to \infty$. Since $\mathcal{B}_1$ is a compact set in the $C^1$-topology, there exists a convergent subsequence (without restriction of generality $(V_k)$ itself) with limit $\hat{V} \in C^1$. Now let $u_k^*$ and $\hat{u}^*$ denote the material derivatives in directions $V_k$ and $\hat{V}$ respectively. From Theorem 4.2 or 4.4, respectively, we obtain for $c(V_k) \to \infty$ the convergence $\| u_k^* \|_{L^2(\Gamma_M)} \to 0$ and due to stable dependence of the material derivative on $V$, this implies $\| \hat{u}^* \|_{L^2(\Gamma_M)} = 0$. Since the limit variation $\hat{V}$ satisfies, the assumptions of Theorem 4.2 or 4.4, respectively, this is a contradiction. Hence, the constant

$$c_0 := \sup_{V \in \mathcal{B}_1} c(V)$$

is finite, which implies the assertion.

\[ \square \]

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References


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