Dynamics of Multibody Systems: Conventional and Graph-Theoretic Approaches

SD 652
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Summary of Course:

- Review of kinematics and dynamics
- Conventional multibody dynamics
- Basics of graph-theoretic (G-T) modelling
- G-T modelling of multibody systems
- Advanced topics
Multibody Mechanical Systems

- a collection of rigid and flexible bodies connected by joints, e.g.

Serial Robots

Parallel Robots (PKMs)
– Walking robots:

http://real.uwaterloo.ca/~robot
– Mechanisms and machinery:

Spatial slider-crank mechanism
- Vehicles (road, rail, aerospace):

Lola World Sports Car
Multibody system dynamics: given only a description of the system as input, formulate the kinematic and dynamic equations needed to determine the system response.

Planar slider-crank mechanism (f=1DOF)
– Coordinates:
  • Must be defined *a priori*
  • Selection affects the number and nature of equations
  • Absolute coordinates:
    – Position and orientation of every body in system, e.g.
      \[
      \mathbf{q} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3, y_3, \theta_3]^T
      \]
    – Easy to formulate equations automatically
    – Very large systems of equations
  • Joint (relative) coordinates:
    – Correspond to joints in system, e.g.
      \[
      \mathbf{q} = [\theta, \beta, s]^T
      \]
    – Fewer in number (minimum for open-loop systems)
    – Requires more topological accounting
– Kinematic Analysis:

• One prescribed motion per dof, e.g. if \( \theta = f(t) \) for slider-crank, solve the kinematic constraint equations for \( \mathbf{q}(t) = [\theta, \beta, s]^T \):

\[
\Phi(\mathbf{q}, t) = \begin{bmatrix} L_1 \cos \theta + L_2 \sin \beta - s \\ L_1 \sin \theta - L_2 \cos \beta \\ \theta - f(t) \end{bmatrix} = 0
\]

• For velocities, solve:

\[
\Phi_{\mathbf{q}} \dot{\mathbf{q}} = -\Phi_t \quad \Rightarrow \quad \Phi_{\mathbf{q}} = \begin{bmatrix} -L_1 \sin \theta & L_2 \cos \beta & -1 \\ L_1 \cos \theta & L_2 \sin \beta & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
– Dynamic equations from Newton-Euler, or from the Principle of Virtual Work:

\[
\delta W = \sum_{n_T} T^T \delta \theta + \sum_{n_F} F^T \delta \mathbf{r} + \sum_{n_{RB}} \delta W_{RB} + \sum_{n_{FB}} \delta W_{FB} = 0
\]

\[
\delta W_{RB} = -ma^T \delta \mathbf{r} - (I\dot{\omega} + \omega \times I\omega)^T \delta \theta
\]

\[
\delta W_{FB} = \int_V \delta \mathbf{r}^T (f_b - \rho \mathbf{a}) dV - \int_V \delta \mathbf{e}^T \sigma dV
\]

– Expressing all variables in terms of \( \mathbf{q} \):

\[
\delta W = Q^T \delta \mathbf{q} = 0
\]

– Set the \( n \) generalized forces \( \mathbf{Q} = 0 \):

\[
\mathbf{M} \ddot{\mathbf{q}} + \mathbf{\Phi}_q^T \lambda = \mathbf{F}
\]
– Where, for the planar slider-crank:

\[
\mathbf{M} = \begin{bmatrix}
m_1 L_1^2 / 3 & 0 & 0 \\
0 & m_2 L_2^2 / 3 & -m_2 L_2 \cos \beta / 2 \\
0 & -m_2 L_2 \cos \beta / 2 & m_2 + m_3
\end{bmatrix}
\]

\[
\mathbf{F} = \begin{bmatrix}
0, 0, F - m_2 L_2 \dot{\beta}^2 \sin \beta / 2
\end{bmatrix}^T
\]

\[
\mathbf{\Phi}_q = \begin{bmatrix}
-L_1 \sin \theta & L_2 \cos \beta & -1 \\
L_1 \cos \theta & L_2 \sin \beta & 0
\end{bmatrix}
\]

- A dynamic simulation is obtained by solving the \( n+m \) differential-algebraic equations for \( \mathbf{q}(t) \) and \( \lambda(t) \).
Dynamic equations for 3-link serial robot:

- Evaluation of mass matrix: 27+, 53x, 12 function
Conventional Methods for Multibody Systems

- Based on absolute coordinates
- Large systems of nonlinear DAEs
- Numerical data, not symbolic equations
- Commercial software:
  - Working Model
  - ADAMS
- Automated modelling and simulation
4th-year (senior) design projects:

Hexplorer 6-legged walking robot
Hexplorer 3-DOF leg in extended position
ADAMS simulation of walking maneuver
SAE mini-Baja vehicle (2002 competition):

Winner of ADAMS modelling award

Mechatronic multibody system dynamics

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Research into vehicle stability:

Grapple Skidder (Timberjack Inc)
Research into vehicle stability:

ADAMS simulation of roll-over

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Research into mechanisms and machinery:

6-bar mechanism designs
Research into vehicle suspension design:

Lola World Sports Car (Multimatic Inc)
– ADAMS model of four-post test:
– no chassis flexibility or joint compliance
– rear left-hand suspension:
– optimization results for “heave test”:

<table>
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<tr>
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<th>2</th>
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<tbody>
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<td>7.3%</td>
<td>6.3%</td>
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<td>Method:</td>
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<td>GA</td>
<td>FR</td>
<td>GA</td>
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<td>Optimization of:</td>
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<td>Dampers</td>
<td>Springs &amp; Dampers</td>
<td>Springs &amp; Dampers</td>
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<td>Iterations:</td>
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<td>21</td>
<td>7</td>
<td>29</td>
</tr>
</tbody>
</table>

– gradient-based method gets trapped in a local minimum, while the hybrid GA explores a wider region of the design space.
Research into biomechanics:

- Investigation of metabolic energy consumption for normal and prosthetic gaits.
Research into biomechanics:

- Forward dynamic simulation, realistic friction
Research into biomechanics:

- Forward dynamic simulation, low friction
Beowulf parallel computing cluster:
Modelling using Linear Graph Theory

- Origins in Koningsburg, Prussia, 1732:

The Town of Koningsburg
Topology = 4 land masses connected by 7 bridges:

Leonhard Euler’s sketch of Koningsburg topology
Euler’s “linear graph” representation of topology:

If there are more than 2 nodes with odd valence, then an “Eulerian path” cannot exist, Euler (1732)
“Graph-Theoretic Modelling” (GTM)

- Component models from measurements

- Edge = element, Nodes = connection points

- Through variable $\tau = \text{current} \ (i)$

- Across variable $\alpha = \text{voltage} \ (v)$

- Constitutive equations, e.g. $v = L \frac{di}{dt}$
System model from assembled components:

- e constitutive equations (linear or nonlinear)
- e linear topological equations from system graph
- primary variables determined by tree selection

Electrical Circuit and Linear Graph

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– Linear graph (nodes=frames, edges=elements):

– Vector variables: $\tau = (\underline{F}, \underline{T})$, $\alpha = (\underline{r}, \underline{\theta})$

– Cutsets = dynamic equilibrium for a subsystem

– Circuits = summation of vector displacements around closed kinematic chains
Advantages of G-T modelling approach:

- Very systematic
- Amenable to computer implementation
- Leads to efficient systems of equations
- Symbolic computer programming can be used
- Suitable for real-time simulation, e.g. for virtual reality or hardware-in-loop experiments
- Applicable to multiple physical domains
- Unifying: coordinates may be absolute or joint or other possibilities
– $n$ branch coordinates $\mathbf{q}$ are defined by tree selection:

$$\text{Tree} = \{r_4 - r_7, h_8, h_{10}, s_{11}\} \Rightarrow \mathbf{q} = [\theta_8, \beta_{10}, s_{11}]^T$$

$$\text{Tree} = \{r_4 - r_7, m_1, m_2, m_3\} \Rightarrow \mathbf{q} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3, y_3, \theta_3]^T$$
Mechatronic Multibody Systems
Electrical network + multibody system, coupled by transducer elements:

http://real.uwaterloo.ca/~watflex
Linear graph representation:

- The DC-motors (transducers) have an edge in both the mechanical and electrical sub-graphs.
– the equations for the individual domains are obtained using the electrical and multibody formulations.
– these equations are coupled by the constitutive equations for the transducers, e.g. for the DC-motor:

\[ v = K_v \frac{d\theta}{dt} + Ri + L \frac{di}{dt} \]

\[ T = K_Ti - B \frac{d\theta}{dt} \]

– from a single linear graph representation, the governing equations are automatically derived in symbolic form by a Maple program “DynaFlexPro”. 
Condensator microphone (Hadwich and Pfeiffer, 1995):
– Linear graph representation:

– where, for the moving-plate capacitor,

\[ i_2 = C_2(s) \frac{dv_2}{dt} + \frac{dC_2(s)}{ds} \frac{ds}{dt} v_2 \]

\[ F_2 = \frac{1}{2} \frac{dC_2(s)}{ds} v_2^2 \]
– Selecting the trees shown, and using the current formulation, one obtains 2 equations in terms of $i_2(t)$ and $s(t)$

\[ C_2(s) \frac{dv_2}{dt} + \frac{dC_2(s)}{ds} \frac{ds}{dt}v_2 - i_2 = 0 \]

\[ m_5 \ddot{s} + d_6 \dot{s} + k_6 s + m_5 g - \frac{1}{2} \frac{dC_2(s)}{ds} v_2^2 = 0 \]

– where:

\[ v_2 = -R_1 i_2 - L_3 \frac{di_2}{dt} + E_4(t) \]
Maple Algorithms (DynaFlexPro)

Model Description
(ASCII File - *.dfp)

1) Build Model

Mathematical Model
(Maple Module)

2) Build Equations

Optimized Simulation Code

3) Build Sim Code

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DynaFlexPro output:

\[ \dot{q}, \ddot{q}, \dddot{q} \Rightarrow Q, Q_{\dot{\cdot}}, Q_{\ddot{\cdot}} \]
\[ \Phi, \Phi_q \Rightarrow Kin, Jac \]

\[ M, F, \lambda \Rightarrow M, F, Lambda \]
\[ M\ddot{q} + \Phi_q^T\lambda - F \Rightarrow Dyn \]

Electrical Equations \( \Rightarrow \) E _ Dyn
System Equations \( \Rightarrow \) Sys _ Dyn

DynaFlexPro: PowerTool for Maple 10
(http://www.maplesoft.com/dynaflexpro/)
Spatial slider-crank (Haug, 1989):

- Kinematics, inverse dynamics, forward dynamics.
– ModelBuilder Representation:
– DynaFlexPro Output:

\[
\begin{bmatrix}
-\frac{2}{25} \sin(\theta(t)) - \frac{6}{25} \sin(\eta(t)) \sin(\beta(t)) - \frac{1}{10} \\
\frac{2}{25} \cos(\theta(t)) + \frac{6}{25} \cos(\eta(t)) \sin(\beta(t)) + \frac{3}{25} \\
-\frac{6}{25} \cos(\beta(t)) + s(t) \\
\theta(t) - 2\pi t
\end{bmatrix}
\]

– Kinematic equations for crank driven at constant speed of \(2\pi\) rad/s.
– DynaFlexPro Output:

– Torque required to drive crank at constant speed of $2\pi$ rad/s.
Gough-Stewart PKM (Wang+Gosselin, 2000):
Driving torques (Tsai, 2000) obtained using symbolic and symbolic/numeric approaches:
CPU time (ms, Matlab on 333Mhz Pentium) for 1 inverse dynamic analysis of all 6 driving forces:

<table>
<thead>
<tr>
<th>Approach</th>
<th>Without sparse methods</th>
<th>With sparse methods</th>
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<tbody>
<tr>
<td>Implicit symbolic</td>
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<tr>
<td>Sym/num (dummy J)</td>
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<tr>
<td>Sym/num (dummy $\Phi_{qd}$)</td>
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<tr>
<td>End effector, numeric</td>
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<td>150</td>
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<tr>
<td>End effector, symbolic</td>
<td>240</td>
<td>240</td>
</tr>
</tbody>
</table>
Automatic gate with flexible bar:

![Diagram of automatic gate with flexible bar](image)
– Linear graph of multibody system + induction motor:
– Rotation of input link:
– Current through one rotor inductor:
Advanced Topics:

- Modelling of tires in vehicle dynamics
- Modelling of flexible bodies, e.g. beams
- Modelling of mechatronic systems
- Modelling of contact dynamics
- Modelling using subsystems
Research into contact dynamics:

Special Purpose Dexterous Manipulator
SPDM

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Research into mechanisms and machinery:

Piano Action (Steinway and Sons)
Subsystem modelling of mechatronic systems

- Single link plus revolute joint condensed into subsystem model using generalization of Norton and Thevenin theorems for electrical circuits.
– Combine 2 link subsystems to form a “leg”:
– Assemble leg subsystems to form parallel robot:
Electro-mechanical subsystem modelling:
– PD-controller subsystem model:

– Combining all graph-theoretic subsystem models:
Induction motor model:

\[
\mathbf{L}_S = \begin{bmatrix}
L_S & M_S & M_S \\
M_S & L_S & M_S \\
M_S & M_S & L_S
\end{bmatrix} \quad \mathbf{L}_R = \begin{bmatrix}
L_R & M_R & M_R \\
M_R & L_R & M_R \\
M_R & M_R & L_R
\end{bmatrix}
\]

\[
\mathbf{M}_{SR} = M_{SR} \begin{bmatrix}
\cos(p\theta_r) & \cos(p\theta_r + \frac{2\pi}{3}) & \cos(p\theta_r + \frac{4\pi}{3}) \\
\cos(p\theta_r + \frac{4\pi}{3}) & \cos(p\theta_r) & \cos(p\theta_r + \frac{2\pi}{3}) \\
\cos(p\theta_r + \frac{2\pi}{3}) & \cos(p\theta_r + \frac{4\pi}{3}) & \cos(p\theta_r)
\end{bmatrix}
\]

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Linear graph model of induction motor:

- For mutual inductance (MI) edge $j$

$$v_j = \sum_{k=1}^{p} L_{jk} \frac{di_k}{dt} - \sum_{k=1}^{p} \frac{d}{dt} \left( M_{SRjk} i_k \right)$$

- The motor output torque is:

$$T_{29} = \frac{1}{2p} \sum_{j=10,11,12} \sum_{k=13,14,15} \frac{\partial}{\partial \theta} \left( M_{SRjk} i_j i_k \right)$$
Bondgraph modelling of multibody system:

**FIGURE 9.17.** Double upside-down pendulum system.

*System Dynamics: Modelling and Simulation of Mechatronic Systems*, Karnopp, Margolis, and Rosenberg [2000]
FIGURE 9.18. Bond graph for the double upside-down pendulum system.
2-link manipulator: linear graph versus...
...bondgraph:

\[ -d_1 \cdot \sin(\theta_1) \]

\[ MTF \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \]

\[ d_1 \cdot \cos(\theta_1) \]

\[ MTF \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \]

\[ -l_1 \cdot \sin(\theta_1) \]

\[ MTF \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \]

\[ -d_2 \cdot \sin(\theta_2) \]

\[ MTF \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \]

\[ I \]

\[ \text{Se} \]

\[ \text{GY} \]

\[ K\varphi_1 \]

\[ \text{R} \]

\[ I \]

\[ \text{Se} \]

\[ \text{GY} \]

\[ K\varphi_2 \]

\[ \text{R} \]

\[ I \]

\[ \text{C} \]

\[ \text{Se} \]
Flexible barrier mechanism and bond graph: