

Lecture 9

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Today we wanted to continue talking about the PCP theorem. We actually even started, but we had to stop early as it was snowing!

Our motivation from last week was proving the following theorem

Theorem 1 *There exists $0 < s < 1$ and Σ (where $|\Sigma|$ is final) s.t. $\text{gap-csp}_s(q=2, \Sigma)$ is NP-hard.*

Put differently, there is some NP-Complete language L , and a polynomial algorithm A s.t. when running A on input x yields a set of constraint $A(x) = C_x$ and

- $x \in L \Rightarrow \text{sat}(C_x) = 1$
- $x \notin L \Rightarrow \text{sat}(C_x) < s$

When $q = 2$ it is helpful to think of the structure of the constraints as a graph. We formally define the constraint graph as follows.

Definition 2 *A graph $G = \langle (V, E), \Sigma, \mathcal{C} \rangle$ is called a constraint graph if (V, E) is a graph, Σ is some finite set (The alphabet) and $\mathcal{C} : E \rightarrow \{\phi : \Sigma \times \Sigma \rightarrow \{0, 1\}\}$. Let $a : V \rightarrow \Sigma$ denote some assignment to the vertices of G , the unsatisfaction level of G from a , denote as $\text{unsat}_a(G)$ is defined as*

$$\text{unsat}_a(G) = \Pr_{(u,v)=e \in E} \left[\mathcal{C}(e) \left(a(u), a(v) \right) = 0 \right].$$

The unsatisfaction level of G , denoted by $\text{unsat}(G)$ is defined as

$$\text{unsat}(G) = \min_a \text{unsat}_a(G).$$

The following theorem explains our keen interest in the constraint graphs.

Theorem 3 *Let $G_1 = (V, E)$ denote some graph. There exists a polynomial (trivial) algorithm A s.t. $G = A(G_1)$ is constraint graph, and*

- $G_1 \in 3\text{COL} \Rightarrow \text{unsat}(G) = 0$
- $G_1 \notin 3\text{COL} \Rightarrow \text{unsat}(G) \geq \frac{1}{|E|}$

Proof Let $\Sigma = \{1, 2, 3\}$. The algoirhtm $A(G_1)$ constructs the graph $G = \langle G_1, \Sigma, \mathcal{C} \rangle$ where

$$\forall e \in E, \forall a, b \in \Sigma : \mathcal{C}(e) \left(a, b \right) = \begin{cases} 1 & a \neq b \\ 0 & \text{otherwise} \end{cases}$$

Clearly A is polynomial. Suppose $G_1 \in 3\text{COL}$ then there exists a coloring in 3 colors s.t. every two adjacent vertices are colored differently (not monochromatic), let a denote the assignment of colors to vertices. Clearly $\text{unsat}(G) \leq \text{unsat}_a(G) = 0$ since a is a valid coloring. If there is no such coloring then for all $a : V \rightarrow \Sigma$ there is at least one monochromatic edge, and thus at least one unsatisfied constraint in G , or $\text{unsat}(G) \geq \frac{1}{|E|}$. ■

Corollary 4 *The following problem is NP-Hard. Let $\Sigma = \{1, 2, 3\}$. Given a constraint graph over Σ decide between*

1. $\text{unsat}(G) = 0$

$$2. \text{ unsat}(G) \geq \frac{1}{|E|}.$$

Our goal is to *amplify* the above corollary into an arbitrary gap, namely we'd like to prove the following

Theorem 5 *There exists some Σ and $0 < \gamma < 1$ s.t. given a constraint graph on alphabet Σ it is NP-Hard to decide between*

$$1. \text{ unsat}(G) = 0$$

$$2. \text{ unsat}(G) \geq \gamma.$$

In order to prove that we require the following main theorem

Theorem 6 *There exists Σ_0 s.t. for all Σ there is some $c > 0$ and $0 < \gamma < 1$ and there exists some polynomial algorithm amp s.t. given a constraint graph $G = \langle (V, E), \Sigma, \mathcal{C} \rangle$ the algorithm amp generate a new constraint graph $G' = \langle (V', E'), \Sigma_0, \mathcal{C}' \rangle$ with the following properties*

$$1. |V'| + |E'| \leq c(|V| + |E|)$$

$$2. \text{ unsat}(G) = 0 \Rightarrow \text{unsat}(G') = 0$$

$$3. \text{ unsat}(G) > 0 \Rightarrow \text{unsat}(G') \geq \min\{\gamma, \text{unsat}(G)\}$$

We leave the proof of the main theorem for a future, less snowy, lesson. We advance to proving Thm.

5. **Proof** We've seen a polynomial reduction from the NP-hard language 3COL into a constraint graph $G = \langle (V, E), \Sigma, \mathcal{C} \rangle$ in Thm. 3 that satisfies

$$1. G \in 3COL \Rightarrow \text{unsat}(G) = 0$$

$$2. G \notin 3COL \Rightarrow \text{unsat}(G) \geq \frac{1}{|E|}.$$

Let $G_0 = G$. For every $i \geq 1$ we let $G_i = \text{amp}(G_{i-1})$ It is easy to see by induction the following.

$$1. \text{ size}(G_i) \leq c \cdot \text{size}(G_{i-1}) \leq c^i \text{size}(G_0)$$

$$2. \text{ unsat}(G) = 0 \Rightarrow \text{unsat}(G_i) = 0$$

$$3. \text{ Let } k = \log_2 |E| + 1. \text{ There exists } i \leq k \text{ s.t. } \text{unsat}(G_i) \geq \gamma. \text{ This follows as } \text{unsat}(G_i) \geq 2^i \text{unsat}(G_0) \text{ unless already } \text{unsat}(G_i) \geq \gamma, \text{ and surely for } k, 2^k \text{unsat}(G) = |E| \cdot \frac{1}{|E|} = 1 \geq \gamma$$

We conclude that the reduction $G \rightarrow G_k$ is the required reduction. ■