Problem Set 2

(due December 20)

1. Let G be a d-regular graph with eigenvalues $\lambda_0 \geq \ldots \geq \lambda_{n-1}$. Prove
   (a) $\lambda_0 = d$.
   (b) $\lambda_0 > \lambda_1$ iff the graph is connected.
   (c) $\lambda_{n-1} = -\lambda_0$ iff the graph is bipartite.

2. Let $G_1 = ([n], E_1)$ be a d_1-regular graph and let $G_2 = ([n], E_2)$ be a d_2-regular graph. Define the graph $G_1 + G_2$ to be a graph whose vertices are $[n]$ and whose edges are $E_1 \cup E_2$ (viewed as a multiset, so parallel edges are allowed). Recall that $h(G)$ denotes the edge expansion of G and $\lambda(G)$ denotes the second largest eigenvalue of G in absolute value.
   (a) What can you prove about $h(G_1 + G_2)$ as a function of $h(G_1)$ and $h(G_2)$?
   (b) What can you prove about $\lambda(G_1 + G_2)$ as a function of $\lambda(G_1)$ and $\lambda(G_2)$?

3. Let $G = (V, E)$ be a d-regular graph with eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Let $S \subset V$ be an arbitrary set with density $\mu = \frac{|S|}{n}$, and let d_S be the average degree in the graph induced on S: $G_S = (S, E \cap S \times S)$.
   (a) Prove $\frac{d_S}{d} \geq \mu + \frac{\lambda_{n-1}}{d}(1 - \mu)$.
   (b) Deduce Hoffman's bound: the size of the maximum independent set in G is bounded by $\frac{-n\lambda_{n-1}}{d-\lambda_{n-1}}$.
   (c) Prove that if t is even, then the number of walks $(r_0, \ldots, r_t)$ in G such that for all $i = 0, \ldots, t$ $r_i \in S$ is at least $\mu n \cdot (d_S)^t$. (Hint: if α is a probability vector then Jensen's inequality implies $\sum \alpha_i x_i^t \geq (\sum \alpha_i x_i)^t$).
   (d) Give a lower bound in terms of μ, d, and λ_{n-1} for the probability of a random walk in G (starting from a random vertex) to stay confined to S. Observe that if λ_{n-1}/d is relatively small this approaches μ^{t+1}, the probability of choosing $t+1$ vertices in V independently and having them all fall in S.

   HINT: For some help, you can look at the paper by N. Kahale, "Better Expansion for Ramanujan Graphs" (section 6), but please mention that you do so.

4. Given two graphs G, H define their tensor product $G \otimes H$ to be the graph on vertices $V(G) \times V(H)$ such that (u_1, v_1) is adjacent to (u_2, v_2) iff $(u_1, u_2) \in E(G)$ and $(v_1, v_2) \in E(H)$.
   (a) Let A, B be the adjacency matrices of G, H respectively. Prove that the adjacency matrix of $G \otimes H$ is the matrix $C = A \otimes B$ defined by $C_{ii',jj'} = A_{i,j} \cdot B_{i',j'}$.

(b) The tensor product of two vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ is $z = x \otimes y \in \mathbb{R}^{nm}$ such that $z_{ij} = x_i \cdot y_j$. Prove that for any two matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m}$

$$(A \otimes B)(x \otimes y) = (Ax) \otimes (By).$$

(c) Deduce that the eigenvalues of $G \otimes H$ are

$$\{\, \alpha\beta \mid \alpha \text{ is an eigenvalue of } G, \text{and } \beta \text{ is an eigenvalue of } H \,\}.$$

(d) Let H be an $(D^4, D, \frac{D}{5})$-graph and let H' be an $(D^8, D, \frac{D}{5})$-graph. Define

$$G_0 = G_1 = H^2$$

$$G_{i+1} = (G_{\lfloor \frac{i}{2} \rfloor} \otimes G_{\lceil \frac{i}{2} \rceil})^2 ⓩ H'$$

Estimate the number n_t of vertices in G_t and prove that $\{G_t\}_{t \geq 0}$ is a family of $(n_t, D^2, \frac{2}{5}D^2)$-expanders.

(e) Prove that $\{G_t\}$ is a very explicit family, i.e., that one can compute the neighbor of a given vertex in G_t in time polylogarithmic in the size of G_t.