The Invariant Set Hypothesis:
A New Geometric Framework for the Foundations of
Quantum Theory and the Role Played by Gravity

T.N.Palmer
ECMWF, UK
tim.palmer@ecmwf.int
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Abstract

The Invariant Set Hypothesis proposes that states of physical reality belong to, and are governed by, a non-computable fractal subset \( I \) of state space, invariant under the action of some subordinate deterministic causal dynamics \( D_j \). The Invariant Set Hypothesis is motivated by key results in nonlinear dynamical-systems theory, and black-hole thermodynamics. The elements of a reformulation of quantum theory are developed using two key properties of \( I \): sparseness and self-similarity. Sparseness is used to relate counterfactual states to points \( p \in I \) thus providing a basis for understanding the essential contextuality of quantum physics. Self similarity is used to relate the quantum state to oscillating coarse-grain probability mixtures based on fractal partitions of \( I \), thus providing the basis for understanding the notion of quantum coherence. Combining these, an entirely analysis is given of the standard “mysteries” of quantum theory: superposition, nonlocality, measurement, emergence of classicality, the ontology of uncertainty and so on. It is proposed that gravity plays a key role in generating the fractal geometry of \( I \). Since quantum theory does not itself recognise the existence of such a state-space geometry, the results here suggest that attempts to formulate unified theories of physics within a quantum theoretic framework are misguided; rather, a successful quantum theory of gravity should unify the causal non-euclidean geometry of space time with the atemporal fractal geometry of state space.
1 Introduction

Many physicists take the view that our understanding of the foundations of quantum physics remains profoundly unsatisfactory, with some suggesting further that this may be hindering progress towards a unified description of physics. The purpose of this paper is to propose a new hypothesis about the nature of physical reality at its most primitive level, to use it to recast key foundational elements of quantum theory, and thereby to make some new proposals on the potential role of gravity in quantum physics.

The Invariant Set Hypothesis is framed in terms of invariance, a concept that forms the very bedrock of physics, and conjectures that states of physical reality are defined by a fractal geometry $I$, embedded in state space and invariant under the action of some subordinate dynamics. The hypothesis is motivated by two concepts that would not have been known to the founding fathers of quantum theory: the generic existence of invariant fractal subsets of state space for certain nonlinear dynamical systems, and the notion that the irreversible laws of thermodynamics are fundamental rather than phenomenological in describing the physics of extreme gravitational systems.

Although quantum theory is unsurpassed in terms of its agreement with experimental data, it is suggested that the Invariant Set Hypothesis provides a geometric framework for a deeper understanding of the foundations of quantum physics than can be provided by quantum theory itself. For example, as discussed in the body of this paper, the new perspective appears to reconcile many of Einstein’s views about the incompleteness of quantum theory with those of the standard Copenhagen Interpretation which emphasises the role of the observer as a partner in the very concept of reality.

Section 2 provides physical motivation for the Invariant Set Hypothesis as a fundamental new postulate in physics. Specifically, Section 2.1 reviews relevant aspects of nonlinear dynamical systems theory, whilst Section 2.2 discusses the state-space flow associated with the contents of the idealised “Hawking Box”, a massive container holding enough matter to form one or more black holes. Here it is argued that the dynamical evolution of small volumes in the state space of the Hawking Box will plausibly asymptote to the types of zero-volume invariant sets discussed in Section 2.1. These ideas are combined to formulate the Invariant Set Hypothesis, discussed in Section 3.

Section 4, the heart of the paper, discusses quantum theory based on an assumption of the correctness of the Invariant Set Hypothesis. Using binary
partitions of $I$, interpreted as elemental observables, the quantum state vector for points of physical reality is straightforwardly interpreted as a probability mixture (Section 4.1). In terms of this, the sparseness of fractal invariant sets in state space provides the basis for understanding why any realistic interpretation of quantum theory must be contextual, as required by the Bell-Kocken-Specker theorem (Section 4.2); essentially, dynamically unconstrained counterfactual states are associated with points of unreality $p \notin I$. This in turn leads to a new perspective on the abstract Hilbert space formulation of quantum theory: the abstract algebraic properties of probability mixtures are used to “continue” the definition of state for points not belonging to $I$. The Gaussian integers, which inherit the algebraic properties of rational integers but cannot be used to describe or compare quantities of physical objects, are discussed by way of analogy. Thus it can be understood why quantum theory describes states of sub-systems in a mathematically consistent albeit abstract way, irrespective of whether the interactions of these sub-systems with the rest of the universe are real or counterfactual. Since $I$ is non-computable, the uncertainty about whether a quantum state represents a probability mixture or not is ultimately linked to mathematical undecidability; this is the fundamental ontology for the uncertainty principle.

State-space magnification of $I$’s self-similar structure by $D_I$’s positive-exponent Lyapunov vectors generates temporal periodicity in these probability mixtures; this is interpreted as the dynamic origin of quantum coherence (Section 4.3). The formalism is further developed to describe probability measures in physical three-space, thus linking to the standard complex Hilbert Space concepts used in quantum theory (Section 4.4).

Combining these perspectives, a comprehensible account of many of the standard “mysteries” of quantum theory (e.g. superposition, nonlocality, the measurement problem, free will, contextuality, emergence of classicality etc) are outlined in Section 5.

Concluding remarks are given in Section 6.

General relativity theory reveals that the role of gravity in classical physics can be described by causal non-euclidean space-time geometry. Here it is suggested that the role of gravity in quantum physics is additionally described by atemporal fractal state-space geometry. This suggests that a successful theory of quantum gravity should unify such space-time and state-space geometries. In particular, attempts to formulate unified theories of physics within a conventional quantum theoretic setting may be misguided.
2 Motivations for the Invariant Set Hypothesis

The remarks in this section provide motivation for the Invariant Set Hypothesis, to be defined in Section 3. There are two different elements of this motivation.

2.1 Dynamically Invariant Sets

Since a key objective of this paper is to develop a new perspective on the role of gravity in quantum theory, it is appropriate to begin discussion with the seminal work of Poincaré, and Birkhoff who found that the motion of as few as three gravitationally-bound Newtonian particles is chaotic, i.e., with aperiodic evolution in a bounded domain of state space. By considering entire cosmologies such as the Mixmaster solution (Misner et al., 1973), it appears general relativity also admits chaotic space-time dynamics. The use of geometric concepts based on the type of invariant set to be considered below appears to be essential for a relativistically-invariant definition of chaos (Cornish, 1997).

Over a half century after Poincaré, Lorenz (1963) proposed a very different type of chaotic motion, associated with forced nonlinear dissipative dynamical systems \( \mathbf{X} = f(\mathbf{X}) \). In contrast with Hamiltonian systems, the states \( \mathbf{X}(t) \) of such systems evolve asymptotically to fractionally-dimensioned (fractal) attractors. If \( \mathbf{X} \) is initialised on an attractor, \( \mathbf{X} \) stays on it forever; the attractor is a dynamically-invariant subset of state space.

Fractal attractors reveal some of the most beguiling of geometries known to physics, and form the basis of discussion in this paper. However, the dynamical systems which generate these geometries are usually considered phenomenological rather than fundamental, since they are explicitly dissipative. In Section 2.2, a more fundamental role for these geometries will be proposed by considering their potential relevance in an extreme general relativistic context.

Such a dynamically-invariant attractor is formed from the asymptotic evolution of a volume \( V(t) \) in state space under the action of the dissipative \( (\nabla \cdot \mathbf{X} < 0) \) dynamics. In the case of the Lorenz (1963) system, \( \nabla \cdot \mathbf{X} \) is constant over state space, in other systems (e.g., Rössler, 1976) \( \nabla \cdot \mathbf{X} \) is flow dependent but negative overall. As discussed in Section 2.2, we envisage systems of the latter type, with regions of state space with \( \nabla \cdot \mathbf{X} < 0 \), corresponding to states containing black holes.
We will need some of the following properties of chaotic invariant sets in the discussions in Sections 3-5.

The invariant sets under consideration have zero measure and are nowhere dense in the Euclidean state space in which they are embedded. On this basis, a perturbation which is random with respect to the continuum measure of state space will almost certainly map a point on the invariant set, off it. As an example, consider the classical Cantor set. A point on the Cantor set can be represented on the interval $[0,1]$ by an exceptional fraction $n_c$ with no digit “1” in its base-3 representation. By contrast a random perturbation can be represented on the interval by a normal (Hardy and Wright, 1979) fraction $n$, whose base-3 representation has equal frequencies of the digits “0”, “1” and “2”. Almost certainly the number $n + n_c$ will contain “1” digits, and hence cannot represent a point on the Cantor set. The sparseness of fractal invariant sets is central to a new perspective on counterfactual reasoning in quantum theory, discussed in Section 4.

Using the $p$-adic metric (Khrennikov, 1997), these invariant sets can be shown to be metric spaces and it is therefore possible to talk about neighbourhoods on the invariant set.

Fractal invariant sets are non-algorithmic. For example, Dube (1993) has shown that the invariant sets of iterated function systems emulate the non-halting states of Turing machines, and undecidable problems in the classical theory of computation have a corresponding geometric interpretation. For example, the Post Correspondence Problem is equivalent to asking whether a given line intersects the invariant set of an iterated function system. More generally, Blum et al (1998) have shown that if an invariant set has fractal dimension, it cannot be a halting set. Penrose (2004) has given arguments for why non-computability may lie at the heart of fundamental physics, providing another motivation for the arguments developed in the heart of this paper.

Invariant sets can be constructed from time series of experimental data, using the Takens Embedding Theorem (Takens, 1981). Suppose the time series of some component of $X$ is measured every $\tau$ units of time. Then the invariant set can be reconstructed from a sufficiently long time series of this component. An important conceptual point relevant to the discussion on free variables below is that it doesn’t matter which specific component of the state vector is used. A sufficiently long time series of even an energetically unimportant or otherwise seemingly irrelevant component of the state vector can be used to construct the invariant set.
One key technique to represent the evolution of states on an invariant set is that of symbolic dynamics (Lind and Marcus, 1995). Consider a bivalent partition $\Pi$ of $I$ based on disjoint subsets $A, B \subseteq I$ such that all points of $I$ either belong to $A$ or to $B$, and a sequence eg. $AABABB...$ where the $n$th member of the sequence represents the subset in which the state $X$ belongs at the $n$th iterate of the dynamical evolution operator. Then the same sequence $AABABB...$ can symbolically represent $X$ at the first iterate, and dynamical evolution is effected by a simple shift of the sequence one place to the left, relative to the radix point. In the case of so-called generating partitions, this symbolic representation is homeomorphic to the original dynamics.

For the situations considered here, “$A$” and “$B$” points will be distributed with fractal measure on the invariant set. That is to say, any neighbourhood $N(I)$ will contain both “$A$” and “$B$” points. On the other hand, as discussed below, the probability distribution of “$A$” and “$B$” points with respect to a certain fixed coarse-graining of $N(I)$ is itself well defined.

An important property of fractal sets is self-similarity. Readers may be familiar with animations which zoom into the Mandelbrot set revealing periodicity in the intricate fractal structure. Below we consider the positive-exponent Lyapunov vectors of the dynamics associated with the invariant set as providing the dynamic “zoom”, thereby generates periodicity in coarse-grain probability measures with respect to the fractal partition. As discussed in Section 4, this will provides the dynamical origin of the oscillatory nature of the quantum wavefunction from the perspective of the Invariant Set Hypothesis.

A key conceptual component of the perspective pursued below, is that the geometry of the invariant set should be considered as more primitive than the differential equations whose asymptotic behaviour generates the invariant set. This is not normally the perspective used in dynamical systems theory where the difference or differential equations are primary. However, in this respect it is worth commenting on some of the global state-space approaches to defining fractal invariant sets. In studying the structure of chaotic attractors in 3-dimensional state space, Birman and Williams (1983) focussed on the knottedness of the assocciated unstable periodic orbits. For example, the knots describing the Lorenz (1963) attractor are prime, fibered with non-negative signature. In higher dimensions, studies have been made characterising fractal invariant sets using simplicial spaces and corresponding homology invariants (Sciamarella and Mindlin, 1999).
In closing this Section, it is worth noting that the very subject of topology arose in significant measure from Poincaré’s attempts to understand and characterise the behaviour of the gravitational 3-body problem, in the absence of any general analytic solution to the problem. Our approach to quantum physics is not dissimilar.

2.2 The Hawking Box

The last section began and ended discussing Poincaré’s work on gravitational systems. Here we discuss the role of an extreme gravitational system which, it is claimed, is pivotal in generating the fractal properties of the invariant set studied in this paper.

Conventional physics is formulated in terms of Hamiltonian dynamics, and the state-space flow is consequently incompressible: $\nabla \cdot \dot{X} = 0$. However, consider a system big enough that a black hole could potentially form from the collapse of matter comprising the system. How can we characterise the asymptotic state space flow of such a system? Is it Hamiltonian?

This has been the topic of considerable debate, especially between two of the leading experts in gravitation theory (Hawking and Penrose, 2000). The debate hinges around a thought experiment in which some vast but gravitationally isolated system is placed in a hypothetical box with reflecting walls - the Hawking Box. What is the asymptotic state-space flow of the Hawking Box?

Two of the most important results in 20th Century theoretical physics are relevant here: the proof that space-time singularities formed by gravitational collapse are generic (Penrose, 1965), and the quantum field theoretic calculation that black holes formed as a result of such gravitational collapse have precise thermodynamic properties (Hawking, 1975). The second law of black-hole thermodynamics describes the loss of information as matter collapses to a black hole, and subsequently evaporates as thermal radiation. As a result of the second law, Penrose (2000) argues that the state space flow of the Hawking Box must be convergent $\nabla \cdot \dot{X} < 0$ in regions of state space containing one or more black holes.

However, Penrose (2000) also argues that shrinking of state-space volumes contradicts Liouville’s theorem and concludes that there must be compensating regions of phase-space divergence, thus motivating Penrose’s (2004) gravitationally-induced Objective Reduction mechanism for quantum state-vector collapse.
A thorough review of the Liouville theorem has been given recently by Ehrendorfer (2006). Liouville’s original paper (Liouville 1838) concerns a result on the material derivative of the Jacobian of the mapping between a solution of a differential equation and its initial state and is not specific to Hamiltonian systems. In a simple form, the Liouville equation for the probability density function $\rho$ of the state vector $X$ is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \dot{X}) = 0$$

(1.1)

reminiscent of the Newtonian mass continuity equation in physical space. If the system is Hamiltonian so that $\nabla \cdot \dot{X} = 0$, then co-moving volumes are conserved in state-space. On the other hand, even for systems where the state-space flow is compressible, the Liouville equation (1.1) guarantees conservation of probability. That is to say, Liouville’s theorem in its general (i.e. non-Hamiltonian) form does not require co-moving volumes in state space to be conserved. If the existence of flow convergence due to black hole formation and evaporation is being assumed, there is no requirement to use the more restrictive volume-preserving form of the Liouville equation. As such, it is concluded that Liouville’s theorem does not itself imply the need for an Objective Reduction mechanism.

Hence, suppose, consistent with Penrose, phase-space volumes shrink in those parts of state space containing black holes, but without any corresponding divergence due to Objective Reduction. What are the consequences?

The situation is analogous to the dissipative dynamical systems approach in the previous section with state-dependent $\nabla \cdot \dot{X} = 0$. It can therefore be supposed that volumes $V(t)$ will shrink to zero and asymptote onto one of the following: a fixed point, a limit cycle, or a fractal attractor. Since fractal attractors are generic, we will assume that the zero-volume asymptotic limit is a fractal set.

Finally, note that we could imagine the Hawking Box without boundary, since it is observationally possible that the observed universe may have compact spatial topology. This raises the notion that states of physical reality may lie on a fractal invariant set, a notion formalised in the next section.
3 The Invariant Set Hypothesis

This last section motivates what will be called “The Invariant Set Hypothesis”. This hypothesis posits the existence of a fractionally-dimensioned subset $I$ of the state space of the physical world. $I$ is an invariant set for some locally-causal deterministic dynamical system $D_I$. That is, points on $I$, hereafter referred to as world states, remain on $I$ under the action of $D_I$.

A crucial conceptual element of the Invariant Set Hypothesis is that $I$ is to be considered more primitive than $D_I$. Given $I$, $D_I$ is merely the operator which translates points (ie world states) along trajectories of $I$. That is, $D_I(t)$ maps some point $p \in I$ a parameter distance $t$ along a trajectory of $I$, and $D_I$ is undefined at points $\not\in I$. This contrasts with the more familiar situation where a dynamical system is defined by differential or difference equations $D$ and generates an invariant set $I_D$ from the asymptotic evolution of $D$.

In the situation where $I$ is considered primitive, then if (cf Section 2.1) $I$ is non-computable, then so is $D_I$. This is consistent with Penrose’s claim (eg Penrose, 1989) that at some deep level, the laws of physics - here $D_I$ - should be considered non-algorithmic. We will insist henceforth that $D_I$ is causal (no influences propagating faster than the speed of light). As discussed below, because of the nature of the Invariant Set Hypothesis, this assumption will not violate the Bell theorem. Hence, $D_I$ can be assumed to describe a unified, deterministic and causal description of the “laws of physics”.

Consistent with the notion of $I$ as primitive, the Invariant Set Hypothesis defines states of physical reality to necessarily lie on $I$. That is to say, with respect to the Invariant Set Hypothesis, points $p \not\in I$ do not correspond to states of physical reality; they are literally “unreal”. In an oracle theory which (non-computability notwithstanding) had perfect knowledge of $I$, these points of unreality are an irrelevance. However, for theories (such as quantum theory) from which the intricate structure of $I$ is hidden, the Euclidean space in which $I$ is embedded cannot be ignored. We return to this in Section 4 where one of the key questions we consider is how to represent quantum-theoretic states in a mathematically-consistent way for such points of unreality.

Focussing on the geometry of the invariant set as primitive, introduces a fundamentally atemporal perspective into the formulation of basic physics. Such a perspective is absent in classical physics, in which differential
equations are considered primitive. The search for an atemporal description of physics has been a long-standing one (e.g. Barbour, 1999; Price, 1996). This has ramifications for a new perspective on the emergence of classicality in quantum physics, discussed further in Section 5.

In keeping with conventional hidden-variable theories, we are presuming that $D_j$ is deterministic and locally causal (e.g. ’t Hooft, 2006, for recent discussion on such ideas). What is new about the approach in this paper is the additional requirement that states of these underlying dynamics are presumed to lie precisely on the invariant set $I$ (see Section 5 below in relation to Bell’s theorem). No classical theory requires states to be constrained to invariant sets, even when the theory supports such sets. In this sense, the proposed “hidden-variable model” being proposed here is certainly not classical.

4 Quantum Theory and the Invariant Set Hypothesis

In this section we discuss the perspective brought to quantum theory by the Invariant Set Hypothesis as outlined above. In Section 4.1 we start by proposing that the quantum state vector be interpreted as a probability mixture based on a partition of $I$. In Section 4.2 we discuss how the sparseness of $I$ provides the basis for an understanding of contextuality in quantum theory. This leads to the notion that the quantum state (as defined in Section 4.1) can be “continued” to points $\in I$ retaining only the algebraic properties of probability mixtures. In Section 4.3 it is claimed that the dynamic origin of the oscillatory component of the quantum state vector is associated with the amplification of $I$’s self-similar structure by positive-exponent Lyapunov vectors. In Section 4.3 we attempt to relate these constructions with the standard complex Hilbert Space of quantum theory.

4.1 Probability Mixtures on the Invariant Set

Quantum theoretic descriptions of states of sub-systems fall under the general definition provided by Hardy (2004): the state of a system is defined to be that thing represented by any mathematical object which can be used to predict the probability associated with every measurement that may be performed on the system.

In quantum theory, states are defined in terms of the complex Hilbert Space. Hence, if $|A\rangle$ and $|B\rangle$ are quantum states, then so is $|\psi\rangle = \alpha|A\rangle + \beta|B\rangle$, where $\alpha, \beta \in \mathbb{C}$. It is, of course, one of the great mysteries of quantum theory (some would say the central mystery) as to the physical reality of the superposed state. The Invariant Set Hypothesis provides a simple answer to
this: on the invariant set, and only on it, $|\psi\rangle = \alpha |A\rangle + \beta |B\rangle$ can be interpreted as a straightforward probability mixture of two discrete alternatives.

How are we to describe some bivalent property of a sub-system $Q$ of the world state. Motivated by the symbolic dynamic description of dynamical systems, consider a bivalent partition $\Pi$ of $I$ based on two disjoint subsets $A_0, B_0 \subset I$, such that all points of $I$ either belong to $A_0$ or to $B_0$. When $p \in A_0$, then we will say that $Q$ has the “A” property, and when $p \in B_0$ then $Q$ has the “B” property. More generally, we define an “observable” of a sub-system in terms of a multivalent partition of $I$; the relationship between this and the quantum theoretic definition of observable is discussed below. Henceforth the subscript $Q$ is dropped.

The world state at $p \in I$ lies, and evolves on, some trajectory segment $T(t)$ whose length is parametrised by the time variable $t \in \mathbb{R}$. Consider a neighbourhood $N_\epsilon(p)$ of $I$ in a subspace of state space transverse to $T$. Generically, the partitions considered here are those where the subsets $A$ and $B$ are fractally intertwined, so that as $\epsilon \to 0$, the probability that a randomly chosen point in $N_\epsilon(p)$ belongs to $A$ does not tend to zero or one, but oscillates indefinitely consistent with the self-similar structure of $I$. Since Planck’s constant has the dimension of phase-space area, we fix the dimensional size of the neighbourhood $N_\epsilon(p)$ using Planck’s constant, ie write $\epsilon = \hbar$. As $p(t)$ evolves on $T$, it carries with it the neighbourhood $N_\hbar(p(t)) = N_{\hbar}(t)$ in the subspaces transverse to $T(t)$.

Let us start by defining a probability mixture $P_{\hbar}(A)$, based on a uniform sampling $\{p_i\}$ $1 \leq i \leq M$ $M \gg 1$ of $N_{\hbar}(p)$. That is to say, using $I$’s metric, construct a regular grid $G$ of points $p_i$ (see Figure 1 below). In terms of this the (frequentist) probability that one of these has the $A$ property is

$$P_{\hbar}(A) = \frac{1}{M} \sum_{i=1}^{M} \chi(p_i)$$

(3.1)

where $\chi(p_i) = 1$ if $p_i \in A$, and $\chi(p_i) = 0$ if $p_i \in B$.

Now at first sight, this definition of probability does not seem to be relevant to the notion of probability in any experimental or operational context. For example, if the sample space is a stream of identically prepared particles, the
sample space must be constructed from a single world state on the trajectory segment $T$, and not from multiple world states in the vicinity of $T$.

However, using $D_I$, equation (3.1) can be interpreted in terms of a probability mixture on $T$. Since by construction $p$ and $p_i$ lie on $I$, there is some trajectory segment of $I$ which joins $p$ to $p_i$. Let us suppose the parameter length of this trajectory segment equals $t_i$; ie, moving forwards from $p$ a parameter distance $t_i$ we arrive at $p_i$. Then we can use this construction to define a new partition $\Pi_i$ of $I$: if $q \in I$ belongs to $A$ (respectively $B$) in the partition $\Pi$, then the point obtained by moving backwards from $q$ a parameter distance $t_i$ will be said to belong to the set $A_i$ (respectively $B_i$) in the partition $\Pi_i$. To make this less abstract, let us colour all the “$A$” and “$A_i$” sets red and all the “$B$” and “$B_i$” sets blue. Then if $p_i$ is a red point with respect to $\Pi$, then $p$ is a red point with respect to $\Pi_i$.

In terms of this, the probability (3.1) can therefore equivalently be defined as

$$P_{\pi}(A) = \frac{1}{M} \sum_{i=1}^{M} \chi_i(p)$$

(3.2)

where $\chi_i(p) = 1$ if $p \in A_i$ and $\chi_i(p) = 0$ if $p \in B_i$.

From a metaphysical perspective, there is something quite profoundly different between the interpretations of probability in equations (3.1) and (3.2). Since $t_i$ is likely to correspond to a very long time interval, the sample space in (3.1), can be assumed to be over states corresponding to different eons of the universe (eg Bojowald, 2007; Penrose, 2008); as such equation (3.1) seems reminiscent of an Everettian perspective of quantum theory (see Section 5.8). By contrast, in equation (3.2) the sample space is over different partitions of $I$ evaluated on a given world state, ie relative a single eon of the universe. This latter perspective is much more in accord with what one might think of as “common sense”, though both are mathematically equivalent.

These two perspectives are reminiscent of the Schrödinger and Heisenberg pictures of quantum theory. In the Schrödinger picture, quantum-theoretic observables are considered fixed in time while the state vector evolves. By contrast, in the Heisenberg picture, the observables evolve whilst the state vector is considered fixed. Similarly, in equation (3.1), we take the partition $\Pi$ as fixed and consider a sample space of a set of world states at different times; whilst in equation (3.2) we take the world state as fixed and consider a sample space of different partitions evaluated from different times.
In considering equation (3.2), a key point to make is that properties of the world state at \( p \in T \), as given by the \( \{ \Pi_i \} \), are determined by the properties of the invariant set in a neighbourhood of \( p \) in state space. This is consistent with treating the invariant set as primitive. There is a similarity with the situation in general relativity where the geometry of space time is considered primitive, and the properties of (eg geodesic) motion of a particle at some point \( q \) in space time are determined not only by the values of the metric tensor at \( q \) but also its values in a small neighbourhood of \( q \).

### 4.2 Contextuality and the Ontology of Counterfactual States

The notion that the quantum state can be interpreted as a probability distribution (cf equations (3.1) and (3.2)) appears to run counter to the Bell-Kochen-Specker theorem which clearly asserts that no such realistic interpretation of the quantum state vector is possible. We discuss here why this theorem does not apply under the conditions of the Invariant Set Hypothesis.

In conventional hidden-variable theory, it is assumed that a preparation procedure creates or prepares a sub-system with certain intrinsic properties, and that different measurements reveal different aspects of these properties. The Bell-Kochen-Specker theorem shows that such non-contextual models are inconsistent with quantum theory. On the other hand, by requiring the world state to lie on a sparse invariant set, such a sub-system’s properties may not in fact be independent of the rest of the world state, ie the rest of the universe. A measurement device is part of the rest of the universe. As such, the assumption that a sub-system necessarily has properties independent of the conceivable measurements are made on it, may be wrong; the Invariant Set Hypothesis provides the basis for understanding why hidden-variable theories must be contextual.

Let us discuss this in more detail. When a counterfactual (Vaidman 2007) world is compared with the corresponding real world, some things are presumed to have changed; others, crucially, have not. For example suppose Bob measures the spin of a spin-1/2 particle with his Stern-Gerlach apparatus oriented in the \( z \) direction. What would the measurement outcome have been had he measured at some angle \( \theta \) to the \( z \) direction? To attempt to give meaning to this question, we imagine a counterfactual world where everything, including the spin-1/2 particle, is as it was in the real world, except that the orientation of the Stern-Gerlach apparatus, here represented by the variable \( \theta \), is different. Can these counterfactual worlds be considered elements of physical reality?
Let \( p \in I \) denote the world state at the time Bob measures the spin of the particle. Let \( l_\theta \) denote the line in state space passing through \( p \) and pointing in the direction where \( \theta \) varies, but where the values of all other components of the world state stay fixed. The intersection of \( l_\theta \) with \( I \) at points other than \( p \) defines the set \( \{ \theta_c \} \) of counterfactual values for \( \theta \) allowed by the Invariant Set Hypothesis (i.e., these points of intersection are counterfactual states which are also states of physical reality). Given \( \{ \theta_c \} \), we can define a binary function \( Sp(\theta_c) \in \{ A, B \} \) such that, according to the dynamics, Bob measures spin up if \( Sp(\theta_c) = A \), and spin down if \( Sp(\theta_c) = B \).

However, the sparseness of \( I \) suggests that counterfactual sets such as \( \{ \theta_c \} \) are in fact equal to the empty set. Consider, for example, a two-dimensional set \( I_{\text{Cantor}} \) formed as the Cartesian product of two Cantor sets, whose \((x, y)\) coordinates are pairs of ternary fractions, each with the missing digit “1”. Now consider a line \( l \) passing through the origin and oriented randomly in the sense that the gradient \( k \) of \( l \) is a normal number. If \( l \) were to intersect \( I_{\text{Cantor}} \) at a point with coordinates \((x_c, y_c) \neq (0, 0)\), then both \( x_c \) and \( y_c \) must be exceptional ternary fractions with missing digit “1”. However, this is almost certainly not the case; since \( k \) is normal, then if \( x_c \) has the missing digit “1”, then \( y_c = kx_c \) almost certainly does not.

The emptiness of the counterfactual set, which we here assume, is consistent with the implications of the Bell-Kochen-Specker theorem, that there can be no non-contextual hidden-variable representations of individual quantum systems; if there were a non-contextual hidden variable model, one could vary one of its parameters keeping other variables corresponding to the rest of the world state, fixed.

If quantum theory could “see” the intricate structure of the invariant set, it would “know” whether a particular putative measurement orientation \( \theta \) was counterfactual or not. However, since, by hypothesis, quantum theory is blind to the intricate structure of \( I \), it is unable to discriminate between factual and counterfactual measurement orientations and therefore admits them all. (The alternative of admitting none would make for a rather vacuous theory!). Hence it is often said that quantum theory requires a particle to be prepared for any measurement that could conceivably be performed on it, counterfactual or otherwise. However, this raises a very fundamental question. If we interpret the quantum-theoretic state as a probability mixture for a point on the invariant set (based on a sample space of partitions - cf
equation (3.2)), how can we interpret the quantum-theoretic state associated with counterfactual world states of unreality, not on the invariant set?

Consider the following analogy. The (rational) integers are rudimentary symbols of counting, used for example to express and compare the quantity of apples in piles of apples. As a result of being symbols of counting, these integers have certain algebraic properties: the sum, difference or product of two integers is a third. Based entirely on these algebraic properties, it is possible to extend the notion of integer to the complex plane, e.g. to the Gaussian integers. These algebraic integers are defined by their algebraic properties, and no more have the primitive property of being symbols of counting; no piles of apples contain $1 + 2i$ apples! As long as we are not concerned whether an integer can be used to count piles of apples, then each point $p$ of a Cartesian grid in the complex plane defines an integer. On the other hand, if we are told, as a result of some empirical study at the local fruit and vegetable shop, that the integer at $p$ describes the quantity of apples in some particular pile of Granny Smiths, then we can infer that $p$ must lie on the real axis!

This analogy is useful in arriving at the required generalisation of “state” in a theory blind to the intricate structure of the invariant set. That is to say, suppose we define “state” by $\alpha |A\rangle + \beta |B\rangle$ irrespective of whether $p \in I$ or not. When $p \in I$ (c.f. the real axis for the Gaussian integers), then $\alpha |A\rangle + \beta |B\rangle$ has a primitive definition as a probability mixture of equation (3.2).

However, when $p \notin I$ then it may still be possible to define “state” as $\alpha |A\rangle + \beta |B\rangle$, but where $\alpha |A\rangle + \beta |B\rangle$ is defined purely by the algebraic properties of probability mixtures. However, under such circumstances $\alpha |A\rangle + \beta |B\rangle$ no more has the property of being a probability mixture of $A$ or $B$ states (c.f. the rest of the complex plane for the Gaussian integers). This “continuation off the invariant set” does not contradict Hardy’s definition of state, since if $p \notin I$ then its points are not elements of physical reality, and hence cannot be subject to actual measurement.

It is worth discussing the corresponding situation in classical physics. A classical dynamical system is one defined by a set of deterministic differential equations. As such, there is no requirement in classical physics for states to lie on an invariant set, even if the differential equations support such a set. (Indeed, for systems which have an invariant set, only with probability zero is the corresponding state likely to evolve on it precisely.) As a result, for a classical system, every point in phase space is a point of “physical reality”,

and the counterfactual states discussed above, are as much states of “physical reality” as are the real-world states. Hence, the world of classical physics is perfectly non-contextual, and is not consistent with the Invariant Set Hypothesis. The question of how classicality can emerge from the Invariant Set Hypothesis is discussed in Section 5.6.

4.3 Quantum Wave Dynamics

In Section 4.1 we interpreted the quantum state as a probability mixture on the invariant set obtained by a coarse-graining of state space. It is well known that the Schrödinger equation is closely linked with the Liouville equation for conservation of probability, for a classical Hamiltonian system. However, there is a feature of the Schrödinger equation which singles it out for special consideration: it admits coherent wavelike solutions. We discuss the origin of this coherence from the perspective of the Invariant Set Hypothesis.

As discussed in Section 3, a key element of the Invariant Set Hypothesis is that properties of some world state \( p \in I \) are determined by the geometry of the invariant set in a neighbourhood of \( p \). A key element of the geometry of the invariant set is self similarity. We propose here that the amplification of self-similarity by positive-exponent Lyapunov vectors is the key to the wave nature of the quantum state.

Figure 1a illustrates schematically this proposal, showing a two-dimensional space \((X,Y)\) with a fiducial trajectory segment \( T(t) \) running up the page on the \( Y \) axis. The figure shows trajectories neighbouring \( T \) on \( I \) diverging exponentially away from \( T \) under the influence of a positive exponent Lyapunov vector. The trajectories are coloured red or blue according to some fractal partition \( \Pi \) of \( I \).

Also shown, for different values of \( Y \), are fixed grids \( G(t) \) in neighbourhoods \( N_{fr}(t) \) transverse to \( T \). For simplicity, \( G(t) \) comprises six fixed gridpoints for each \( t \), though the construction is easily generalised, consistent with equation (3.1), to comprise some large finite number \( M \) of grid points. As illustrated, the probability that one of the gridpoints of \( G(t) \) is, say, coloured red, oscillates periodically in time.

The frequency of this oscillation varies with the magnitude of the Lyapunov exponent. Figure 1b is the same as Figure 1a except that the Lyapunov exponent is twice as large. The frequency of the oscillation in probability is similarly twice as large. More generally, if a particular Lyapunov exponent
scales as $E/h$, then the frequency of the corresponding oscillation in the associated coarse-grain probability will similarly scale as $\omega=E/h$, the free-particle solution to the Schrödinger equation. Hence from the perspective of the Invariant Set Hypothesis, the action of the Lyapunov vectors on the self-similar geometry of $I$ in a neighbourhood of $T$ can be considered the origin of quantum coherence.

As discussed in Section 4.1, we can also interpret probabilities in terms of a sample space of partitions $\{\Pi_i\}$ evaluated on $T$ (cf equation (3.2)). For example, the colour at the $i$th grid point (here $1 \leq i \leq 4$) of $G(t)$ is also the colour of the world state at $p(t) \in T$ associated with the partition $\Pi_i$. Hence, the probability of a red value for $p(t) \in T$ with respect to the sample space $\{\Pi_i\}$ is similarly oscillating in time.

It is postulated here that, on the invariant set, the Schrödinger equation is an equation for conservation of probability in regions where dynamical evolution is Hamiltonian (ie in regions not associated with black holes). The complex number structure of the Schrödinger equation arises in part from the oscillatory nature of the probability mixtures discussed above, and in part from the three-dimensional nature of physical space, discussed in Section 4.4 below. Since quantum theory is blind to the intricate structure of the invariant set, the quantum theoretic Schrödinger equation must be formulated in abstract Hilbert space form (cf Section 4.2) using algebraic properties of probability. One algebraic property inherited from the Schrödinger equation’s interpretation as a Liouville equation on the invariant set, is linearity: as an equation for conservation of probability, the Liouville equation, cf equation (1.1), is always linear, even when the underlying dynamics $\dot{X}=f(X)$ are strongly nonlinear. This suggests that attempts to add nonlinear terms (deterministic or stochastic) onto the Schrödinger equation eg during measurement, are misguided.

### 4.4 Towards the Complex Hilbert Space

Here we develop further the relationship between the notion of probability mixture as defined in equation (3.2) and the quantum-theoretic notion of state vector as an element of a complex Hilbert Space.

A key point from Section 4.3 is that with respect to the sample space of partitions $\Pi_i$, the probability $P_{\pi}(A)$ that the world state $p(t) \in I$ has the property “$A$”, is oscillatory in time. Let us consider representing the probabilities $P_{\pi}(A)$ and $P_{\pi}(B) = 1 - P_{\pi}(A)$ with vectors in $\mathbb{R}^2$ ie by
\( \alpha |A\rangle + \beta |B\rangle \) where \( \alpha^2 + \beta^2 = 1 \) \( \alpha, \beta \in \mathbb{R} \), so that \( \alpha^2 = P_\Pi (A) \) and \( \beta^2 = P_\Pi (B) \).

Hence if these probabilities are oscillatory, then so are \( \alpha \) and \( \beta \).

A second key point for the present discussion is that physical space is three dimensional. Now the definition of state used here depends on the chosen partition. Following Hardy’s definition of “state”, as given at the beginning of Section 4, we want to be able to distinguish between states associated with partitions which correspond to different directions in space. As such, then this probabilistic representation in \( \mathbb{R}^2 \) is inadequate, there are simply not enough degrees of freedom.

Let the unit vector \( \hat{n} \) in physical space denote the direction with respect to which the “A” and “B” properties are defined and let the corresponding probability state be \( \alpha (\hat{n}) |A\rangle + \beta (\hat{n}) |B\rangle \). If instead of \( \alpha (\hat{n}), \beta (\hat{n}) \in \mathbb{R} \) we put \( \alpha (\hat{n}), \beta (\hat{n}) \in \mathbb{C} \) (now with \( |\alpha (\hat{n})|^2 + |\beta (\hat{n})|^2 = 1 \)), then we now have enough degrees of freedom that the probability states are distinguishable for different orientations. Moreover, the (surjective) homomorphism between \( SU(2) \) and \( SO(3) \) defines the way in which \( \alpha \) and \( \beta \) in \( \alpha (\hat{n}) |A\rangle + \beta (\hat{n}) |B\rangle \) transform as \( \hat{n} \) varies on \( I \). That is, as \( \hat{n} \) varies under the action of some specific element of \( SO(3) \), the probability mixture at \( p \) transforms under the action of the corresponding element of \( SU(2) \).

On the invariant set, an entangled quantum state is represented, quite straightforwardly, by probability mixtures on correlated sample spaces. Hence, if \( \{A, B\} \) and \( \{C, D\} \) define two fractal partitions of \( I \) then oscillating probability states can be defined for the combined partition based on \( \{A \cap C, A \cap D, B \cap C, B \cap D\} \). The quantum state is

\[
\alpha |A\rangle |C\rangle + \beta |A\rangle |D\rangle + \gamma |B\rangle |C\rangle + \delta |B\rangle |D\rangle
\]

(3.3)

where, for example, \( |\alpha|^2 \) denotes the probability that the world state has both the \( A \) property and the \( C \) property.

More generally, define an “observable” as an intertwined \( N_p \)-element partition \( \{A, B, C, \ldots Z\} \) of \( I \). The corresponding sample space (from which the probability mixtures are defined) is now an \( N_p \)-element string

\( S = \{a_1 a_2 a_3 \ldots \} \) where \( a_i \in \{A, B, C, \ldots Z\} \).
In quantum theory, from which the intricate structure of these partitions of \( I \) is hidden, an observable must be defined in a way which does not depend on whether or not a given point lies on the invariant set. The concept of algebraic abstraction has already been discussed in relation to the notion relating state vector and probability distribution. The abstraction needed to describe observable in the quantum-theoretic situation where is it unknown (and algorithmically unknowable) whether \( p \) belongs to \( I \) or not, is well known: an observable is treated as a Hermitian operator on the abstract Hilbert Space states. Being Hermitian, the \( N_p \) eigenvalues of the observable are real, and, for \( p \in I \), the \( N_p \) eigenstates correspond to probability distributions where it is certain which of the \( N_p \) elements of the partition of \( I \) the world state belongs.

5 A Novel Perspective on the “Mysteries” of Quantum Theory

We put the ideas developed in Section 4 to work, providing a new and comprehensible interpretation of the standard mysteries of quantum theory.

5.1 The Superposed State and Quantum Coherence

In quantum theory, the superposed state characterises the notion of quantum coherence and cannot be interpreted as a probability mixture. In the context of the Invariant Set Hypothesis, coherence arises from the self-similar structure of the invariant set. As such, the superposed quantum state \( a |A\rangle + \beta |B\rangle \) has no fundamental ontological significance; it merely reflects a probability mixture when describing points on the invariant set. Just as points on the invariant set are either in \( A \) or in \( B \); Schrödinger’s cat is alive or dead, and not both.

5.2 The Measurement Problem

The Invariant Set Hypothesis provides a straightforward resolution of the measurement problem. By performing measurements we humans acquire information about the world state. That is, by empirical means, we become more knowledgeable about \( I \) than could ever be gleaned from quantum theory alone. In quantum theory this empirical information is ingested into the description of state through a “collapse of the wavefunction”.

According to the Invariant Set Hypothesis, the notion of superposition has no fundamental ontological significance, hence neither has measurement; there
are no jumps in the world state as a result of measurement since the world state was never in a superposition in the first place.

5.3 The Copenhagen Interpretation vs Einstein Reality

The Invariant Set Hypothesis reconciles Einstein’s views on quantum theory, with those of Bohr, Heisenberg and Pauli, as summarised in the Copenhagen Interpretation.

On the one hand, consistent with Einstein’s view, the Invariant Set Hypothesis indicates that quantum theory is incomplete in the sense that it is blind to the fractal structure of the invariant set and hence $D_I$. Quantum theory only sees the coarse-grain structure of $I$; it is theory saddled within $L^I$ spectacles. With respect to $D_I$, physics is both deterministic (no dice) and locally causal (no spooky effects).

On the other hand, the Invariant Set Hypothesis provides an objective basis for understanding why the observer is a partner in the very concept of reality. From the Invariant Set Hypothesis, it is not meaningful to regard an individual quantum system as having any intrinsic properties independent of the invariant set on which the whole world state evolves. The invariant set is in part characterised by the experiments which inform us humans about it. Hence, the Invariant Set Hypothesis implies that it is not meaningful to regard a quantum particle as having any intrinsic properties independent of the measurements performed on the quantum system. This is one of the key tenets of the Copenhagen Interpretation.

5.4 Delayed Choice Measurement

One of the mysteries of quantum theory is the notion that the state of a quantum sub-system at $t = t_0$ can be influenced by measurements whose characteristics are only determined at $t \gg t_0$. For example, deciding whether to make a “wave measurement” or a “particle measurement” can seemingly be deferred until after the time when the quantum sub-system has to “make up its mind” whether to behave as a wave or as a particle.

This paradox is readily resolved by recalling that the notion of the invariant set is an atemporal one - whether or not a world state lies on the invariant set at some time $t = t_0$ and hence is a point of reality, depends on events both to the (indefinite) future and past of $t_0$. This is effectively another expression of the non-computability of the invariant set.
There is an analogy here with the event horizon of a black hole. Whether a point in space time lies on an event horizon or not may depend on events in the far future of that point.

5.5 Non-locality

As Bell has pointed out (Bell, 1995) the notion that quantum mechanics is not locally causal (ie is nonlocal), depends on treating experimental parameters, such as the orientation of measuring devices, as free variables. Since $D_i$ is considered to be causal, the role of the Invariant Set Hypothesis as a restriction on the existence of free variables becomes a central issue in assessing whether quantum physics is nonlocal.

‘tHooft (2007) relates the notion of a free variable to what is called the Unrestricted Initial State condition, which he describes as having consequences similar to free will, but not clashing with determinism. In motivating the Unrestricted Initial State condition, ‘tHooft says “…we must demand that our model [of nature] gives credible scenarios for a universe for any choice of the initial conditions.”

The Unrestricted Initial State condition is certainly plausible for any physical theory based solely on differential equations, eg the laws of classical physics. However, it manifestly fails in the Invariant Set Hypothesis. In the Invariant Set Hypothesis, the only “credible scenarios” are associated with initial conditions which lie on the invariant set. As discussed above, this is a sufficient restriction to rule out counterfactual states.

Using the language of hidden variable theory, let $P(\lambda|\theta_a,\theta_b)$ denote the probability of some hidden-variable $\lambda$ given EPR measurement orientations $\theta_a$ and $\theta_b$. Conventionally, in hidden-variable theory, one assumes that $P(\lambda|\theta_a,\theta_b) = P(\lambda)$ (eg Weinstein, 2008). However, in terms of the Invariant Set Hypothesis, this assumption of statistical independence is certainly not valid. Over the neighbourhood $N_{\lambda,p}(p)$ one can imagine $P(\lambda)$ to be well approximated by some continuum probability distribution. However, if the invariant set is sufficiently sparse that the counterfactual set is empty, then $P(\lambda|\theta_a,\theta_b) = \delta(\lambda - \lambda_{a,b})$ and therefore not even approximately equal to $P(\lambda)$. Hence, although the dynamics $D_i$ is deterministic and causal, the Invariant Set Hypothesis provides sufficient constraint to prevent spin correlations associated with $D_i$ being constrained by Bell inequalities.
One philosophical objection to any restriction on the notion that experimental parameters are “free”, is that it might appear to make us humans, who fix these parameters, seem no different to automata. However, by the Invariant Set Hypothesis, this is not the case. We humans are conscious beings. As such, we acknowledge as “real” the physical world around us. Hence, by the Invariant Set Hypothesis, we acknowledge the reality of \( I \). That is, we acknowledge the reality of something which is fundamentally non-algorithmic. As stressed by Penrose (1989), no automata would be capable of this!

Another potential objection to the notion that experimental parameters are other than “free”, is that one could set these experimental parameters on some whim, eg the toss of a coin, or the outcome of the Swiss Lottery, or the sixth digit of the frequency of a photon originating in Alpha Centuri. If measurement orientations can depend on outcomes which seem “irrelevant” as far as the evolution of the rest of the universe is concerned, then surely these experimental parameters are for all practical purposes, free? This was an argument John Bell himself used (Bell, 1995).

Here we refer back to the Takens Embedding Theorem - that the entire invariant set can be reconstructed from a sufficiently long timeseries of any component of the state vector, even if this component is energetically unimportant and seemingly irrelevant to the evolution of the rest of the universe. Hence it is inconsistent to conclude from the seemingly whimsical examples above, that measurement orientations can be made dependent on variables which are themselves irrelevant to or independent of the evolution of the rest of the world state.

Theories which constrain the existence or notion of free variables might be classed as “superdeterministic”. However, it is important to distinguish the conventional idea of superdeterminism with that of the Invariant Set Hypothesis. Conventional superdeterminism is an \textit{ad hoc} approach and hence unappealing. For example, superdeterminism presumes that the actual initial state of the universe is the only allowable initial state. Yet, why should this be? By contrast, the restrictions on experimental parameters implied by the Invariant Set Hypothesis are a consequence of invoking a certain type of invariance - dynamical invariance. As is well known, invariance and symmetry are the bedrocks of theoretical physics and therefore not at all \textit{ad hoc}. By contrast with superdeterministic thinking, the Invariant Set Hypothesis provides a sound theoretical basis for constraining what would otherwise free variables.
5.6 Emergence of Classicality

It can be noted that, relative to the Invariant Set Hypothesis, the classical limit cannot be reached by letting $\hbar \to 0$; as discussed, the fractal structure of $I$ persists to all scales, no matter how small. This is consistent with $I$ being non-computable. In this sense the limit $\hbar = 0$ is a singular limit (c.f. Berry, 2002).

How could classical physics emerge from the Invariant Set Hypothesis? One way to answer this is to consider the structure of the invariant set associated with time-averaged states. By the central limit theorem, we know that the invariant measure associated with sufficiently long time-averaged states is Gaussian, and hence not fractal. A similar continuum measure would arise if $I$ was projected into a small dimensional subspace (cf “tracing out” the environmental degrees of freedom to use the language of decoherence theory). With such a smooth measure, the counterfactual set would not be the empty set, indeed in the (Gaussian) limit, a counterfactual set such as $\{\theta\}_c$ in Section 4.2 would be all of $0 \leq \theta \leq 2\pi$ and all the (fractal-based) arguments above would fail. In this (time-averaged or projected) situation, the system would behave as if it were classical.

5.7 Quantum Uncertainty

Using the concept of algebraic abstraction discussed in Section 4.2, quantum theory provides a consistent mathematical definition of state, irrespective of whether that state describes a point on the invariant set or off it. In the former case, the state corresponds to a probability mixture, in the latter case it does not. Quantum theory is blind to the intricate structure of the invariant set, and because the invariant set is uncomputable, quantum theory cannot be supplemented by any algorithm to determine whether a point belongs to the invariant set of not. In this sense, mathematical undecidability, it is claimed is the essential source of uncertainty in quantum theory.

5.8 Quantum Theory and the Multiverse

As has been mentioned, the Invariant Set Hypothesis is profoundly atemporal. Nowhere is this more apparent than in the representation of the quantum state as a probability mixture associated with a coarse-graining of $I$’s fractal structure. As mentioned in Section 4, although the points over which the coarse-graining is performed are close together in state space, they are likely to be very far apart in terms of the parametrised trajectory length (viz time) associated with $D_I(t)$. Indeed from the $t$ perspective, these points
effectively belong to different eons of the universe (see Section 3). As such, the sample space for the coarse-graining in equation (3.1) can be considered to be a multiverse of solutions to the laws of physics. (It is a mute point whether this sample space comprises material universes, or, rather, mathematical solutions which merely have the potential to be materially real). The notion of such a sample space would seem bizarre if we were to take $D_i$ as primitive. However, by treating instead the atemporal geometry $I$ as primitive, this sample space merely describes points of some neighbourhood of $I$, a construct neither problematic nor causally paradoxical. This suggests a potential dynamical comprehensible basis for Everettian interpretations of quantum theory (Deutsch, 1997).

5.9 Quantum Computing

The notion that the evolution of properties of the quantum state depends on the structure of the invariant set in the neighbourhood of a point is not a classical concept. Rather, in classical physics, the evolution $\dot{X} = F(X)$ of a state depends only on the values of the state at a single point in state space. This appears to provide the conceptual basis to understand the power of quantum computing over classical computing.

5.10 The Role of Gravity in Quantum Physics

Gravity has often been suggested as playing a role in quantum theory, principally as a mechanism that induces quantum state vector collapse (Penrose, 2004).

As discussed above, at an ontological level, the Invariant Set Hypothesis does not require superposed states and hence does not require a gravitational collapse mechanism. On the other hand, the order-of-magnitude estimates provided by Penrose (2004), that gravitational processes can be locally significant when a quantum sub-system and a measuring apparatus interact, seem persuasive. Here we would interpret these estimates as supporting the notion that gravity plays a key role in defining the state space geometry of the invariant set. At the extreme level, this has already been discussed in Section 2.2, whereby black hole dynamics provides the basis for the dimensional reduction of the invariant set in state space.

If gravitational processes play a key role in defining the invariant set, and if quantum theory is blind to the intricate structure of the invariant set, then it is misguided to assume (as almost all serious attempts have so far done) that
quantum gravity, more generally “theories of everything”, can be formulated in terms of conventional quantum theory.

General Relativity theory reveals that the role of gravity in classical physics can be understood in terms of its causal effect on space-time geometry. The Invariant Set Hypothesis conjectures that the role of gravity in quantum physics can be understood in terms of its atemporal effect on state-space geometry. As such, a challenge for the future will be to combine the pseudo-Riemannian geometry of space-time, and the fractal geometry of state space. This is a very different perspective on “quantum gravity” than suggested in any existing approaches to the subject.

6 Conclusions

Principles of invariance and symmetry lie at the heart of the foundations of physics. We have introduced a new type of invariance; the Invariant Set Hypothesis subordinates the notion of the differential equation and elevates as primitive the notion of fractal state space geometry in defining the notion of physical reality. It is suggested that this has profound implications for our understanding of quantum theory as discussed at length in the body of this paper.

The Invariant Set Hypothesis is motivated by two quite disparate ideas in physics. Firstly, certain nonlinear dynamical systems have measure-zero, nowhere-dense, self-similar non-computational invariant sets. Secondly, the behaviour of extreme gravitationally bound systems is described by the irreversible laws of thermodynamics at a fundamental rather than phenomenological level.

General relativity has already elevated geometry as a key concept for investigating the causal structure of space time. The Invariant Set Hypothesis similarly elevates geometry as a key concept for understanding the atemporal structure of quantum physics.

In the 1960s, the introduction of global space-time geometric and topological methods, transformed our understanding of gravitational physics in space time (Penrose, 1965). It is proposed that the introduction of global geometric and topological methods in state space, may similarly transform our understanding of quantum physics and the role of gravity in quantum physics. Combining these disparate forms of geometry may provide the missing element needed to advance the search for a unified theory of basic physics.
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1. a) Schematic illustration showing a set of trajectory segments of $I$ diverging exponentially from a central fiducial trajectory $T$, under the action of some positive-exponent Lyapunov vector. This divergence, acting on the self-similar structure of some underlying red/blue fractal partition of $I$, generates an oscillatory probability distribution on a fixed grid $G(t)$ of six gridpoints in neighbourhoods of $I$ transverse to $T$. For example, the probability of "red" varies from bottom to top, as 6/6, 4/6, 2/6, 0/6, 2/6, 4/6, 6/6 and so on. b) As a) but with a Lyapunov exponent twice as large. The frequency of oscillation of probability is similarly twice as large.