Class 4: Mathematical Modeling of Mechanical Systems

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WARNING!

• I claim no originality in all these notes. These are the compilation from various sources for the purpose of delivering lectures. I humbly acknowledge the wonderful help provided by the original sources in this compilation.

• For best results, it is always suggested you read the source material.
Contents

• Basic elements of an mechanical system
  – Translational and rotational
• Equations for the basic elements
• Numerical and its solutions
• Summary
• Unsolved Problems
• References
Types of Mechanical Systems

• Translational Systems
• Rotational Systems

Based on their mode of displacement
Elements of Mechanical System
a. Mass

- The ideal mass element represents a particle of mass, which is the lumped approximation of the mass of a body concentrated at the centre of mass. The mass has two terminals, one is free, attached to motional variable $\nu$ and the other represents the reference.

- In previous figure, $f(t)$ represents the applied force, $x(t)$ represents the displacement, and $M$ represents the mass. Then, in accordance with Newton’s second law,

$$f(t) = Ma(t) = \frac{Mdv(t)}{dt} = \frac{Md^2x(t)}{dt^2}$$

- Where $v(t)$ is velocity and $a(t)$ is acceleration. It is assumed that the mass is rigid at the top connection point and that cannot move relative to the bottom connection point.

$$f(t) = \frac{Md^2x(t)}{dt^2}$$
b. Damper

- Damper is the damping elements and damping is the friction existing in physical systems whenever mechanical system moves on sliding surface. The friction encountered is of many types:
  - Stiction: The force required at startup.
  - Coulomb Friction: The force of sliding friction between dry surfaces. This force is substantially constant.
  - Viscous friction force: The force of friction between moving surfaces separated by viscous fluid or the force between a solid body and a fluid medium. It is linearly proportional to velocity over a certain limited velocity range.
Equation of motion in a damper

• In friction elements, the top connection point can move relative to the bottom connection point. Hence two displacement variables are required to describe the motion of these elements. A physical realization of this phenomenon is the viscous friction associated with oil and so forth. A physical device that is modeled as friction is a shock absorber. The mathematical model of friction is given by

\[ f(t) = B \left( \frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} \right) \]

– where \( B \) is the damping coefficient.
c. Spring

- The final translational mechanical element is a spring. The ideal spring gives the elastic deformation of a body. The defining equation from Hooke’s law, is given by

\[ f(t) = K(x_1(t) - x_2(t)) \]

- Note here that the force developed is directly proportional to the difference in the displacement of one end of the spring relative to the other. These equations apply for the forces and the displacements in the directions shown by the arrowheads in figure.

- If any of the directions are reversed, the sign on that term in the equations must be changed. For these mechanical elements, friction dissipates energy but cannot store it. Both mass and a spring can store energy but cannot dissipate it.
# Mathematical models of translational elements

<table>
<thead>
<tr>
<th>Element</th>
<th>Symbol</th>
<th>Force velocity</th>
<th>Force displacement</th>
<th>Force acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>![Mass Icon]</td>
<td>$F = M \int \frac{dv}{dt}$</td>
<td>$F = M \frac{d^2x}{dt^2}$</td>
<td>$F = Ma$</td>
</tr>
<tr>
<td>Spring (one end fixed)</td>
<td>![Spring Icon]</td>
<td>$F = k \int vdt$</td>
<td>$F = kx$</td>
<td></td>
</tr>
<tr>
<td>Dashpot (one end fixed)</td>
<td>![Dashpot Icon]</td>
<td>$F = Bv$</td>
<td>$F = B \frac{dx}{dt}$</td>
<td></td>
</tr>
<tr>
<td>Spring (Displaystyle in both ends)</td>
<td>![Spring Icon]</td>
<td>$F = k \int (v_1 - v_2)dt$</td>
<td>$F = k(x_1 - x_2)$</td>
<td></td>
</tr>
<tr>
<td>Dashpot (Displaystyle in both ends)</td>
<td>![Dashpot Icon]</td>
<td>$F = B[v_1 - v_2]$</td>
<td>$F = B \frac{d}{dt}(x_1 - x_2)$</td>
<td></td>
</tr>
</tbody>
</table>
Numerical No.1

• Find the transfer function of the mechanical translational system as in Figure using equations of balance as described in the chapter
Solution to Numerical No.1

- The equations of motion is given by,

\[ F(t) = M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + kx(t) \]

- Depending on the nomination of input and output, the transfer function can be derived. For example, in simple terms f(t) is input and x(t) may be designated as output.

\[ F(s) = Ms^2X(s) + BsX(s) + kX(s) \]

i.e. \( F(s) = (Ms^2 + Bs + k)X(s) \)

Transfer Function

\[ G(s) = \frac{X(s)}{F(s)} \]
Mechanical Rotational Systems

\[ T = \frac{J d^2 \theta}{dt^2} = \frac{J d\omega}{dt^2} \]

\[ T = K(\theta_1 - \theta_2) \]

\[ T = B\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = B(\omega_1 - \omega_2) \]
Mathematical models of rotational elements

<table>
<thead>
<tr>
<th>Element</th>
<th>Symbol</th>
<th>Torque-angular velocity</th>
<th>Torque-angular displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational</td>
<td>![Rotational Symbol]</td>
<td>$T = J \frac{d\omega}{d\theta}$</td>
<td>$T = J \frac{d^2\theta}{dt^2}$</td>
</tr>
<tr>
<td>Dashpot</td>
<td>![Dashpot Symbol]</td>
<td>$T = B\omega$</td>
<td>$T = B \frac{d\theta}{dt}$</td>
</tr>
<tr>
<td>Spring</td>
<td>![Spring Symbol]</td>
<td>$T = k \int \omega dt$</td>
<td>$T = B\theta$</td>
</tr>
<tr>
<td>Dash pot with both ends free</td>
<td>![Dash Pot with Both Ends Symbol]</td>
<td>$T = B(\omega_1 - \omega_2)$</td>
<td>$T = B \frac{d}{dt}(\theta_1 - \theta_2)$</td>
</tr>
<tr>
<td>Spring with both ends free</td>
<td>![Spring with Both Ends Symbol]</td>
<td>$T = k \int (\omega_1 - \omega_2) dt$</td>
<td>$T = k(\theta_1 - \theta_2)$</td>
</tr>
</tbody>
</table>
Numerical No. 2

• Find the transfer function $X_2(s)/F(s)$ for the system in the figure below:
Numerical No.3

• Find the transfer function of the mechanical translational system \( G(s) = \frac{X_2(s)}{F(s)} \) in figure.
Numerical No.4

• For the rotational mechanical systems shown in figure (a) and (b), write the equations of motion.
Summary

• Identify the basic elements (translational and/or rotational) in the given system.
• Write equations for the basic elements using governing laws.
• Relate them in terms of specified input and output
• If necessary, do the computation in Laplace domain
References

- MIT OCW material: Lecturer: Penmen Gohari
  amongst others...
And, before we break...

• I am not a has been. I am a will be
  – Lauren Becall

Thanks for listening...