Trends and Perspectives

An Assessment of Geometric Methods in Trajectory Synthesis for Shape-Creating Manufacturing Operations

Radha Sarma, Dept. of Mechanical, Industrial, and Manufacturing Engineering, The University of Toledo, Toledo, Ohio. E-mail: rsarma@eng.utoledo.edu

Abstract

This paper presents an assessment of the steps involved in trajectory synthesis (for example, trajectory layout, trajectory generation, trajectory spacing, trajectory postprocessing, and trajectory verification) for shape-creating manufacturing operations such as rapid prototyping, milling, and electrical discharge machining. The rationale for this paper is that trajectory plays an important role in determining productivity, as explained below.

Shape-creating manufacturing operations are those that make use of a "tool" that operates on a "workpiece" to create the desired shape. The tool operates on the workpiece as dictated by the trajectory of the tool, resulting in the manufactured shape. Often, the geometric and functional properties of the end product are dependent on the trajectory of the tool. Moreover, the trajectory dictates the accuracy of the end product and the time taken to manufacture the product. Recognizing the important contribution of the trajectory in shape-creating operations, this paper therefore focuses on trajectory synthesis.

Keywords: Manufacturing Process, Path Generation, Numerical Control, Machining, Shape Creation

Introduction

Traditional manufacturing processes can be broadly classified based on whether one is primarily interested in the geometric properties of the manufactured product (as in milling or rapid prototyping) or the material properties of the manufactured product (as in heat treatment or chemical treatment). Most often, one is interested in the overall properties of the manufactured product that allow it to perform the function it is designed for. Of relevance to this paper are methods that focus on obtaining pre-specified geometric properties, such as shape joining (joining objects along common surfaces, curves, or points), shape generating (generating an object by adding or subtracting material), shape transforming (transforming the shape of objects without adding/subtracting material), and shape duplication (duplicating the shape of an object) operations.

These manufacturing operations are collectively referred to as shape-creating manufacturing operations (SCMOs) and account for a significant volume of current manufacturing operations. Figure 1 demonstrates a classification (showing a few examples of traditional processes) of manufacturing operations with focus on geometric properties that expands on a classification by Dhande and Karunakaran.17 Among the SCMOs, geometry-intensive trajectory synthesis plays a significant role in shape joining (determining the positions and orientations of the weld rod in arc welding) and shape generating (determining the positions and orientations of the milling tool in milling) operations, a relatively minor role in shape transforming (determining the path of the dies in stamping machines) operations, and an indirect role in shape duplication (in using electrodes or dies) operations.

Currently, SCMOs are commonly achieved by means of numerically controlled machine tools. Numerical control (NC) is the automated operation of machine tools by a series of coded instructions

![Figure 1](classification-of-manufacturing-processes.png)
that cause a tool to move relative to a workpiece. The tool may have \( n \) degrees of freedom relative to the workpiece (referred to as an \( n \)-axis machine). The tool may have point (milling of free-form surfaces), curve (wire electrical discharge machining [EDM]), or surface contact (stamping) with the desired shape.

Note that the terms “tool” (for example, an end mill in NC milling, a laser beam in laser machining, a nozzle in fused deposition manufacturing, or a paint jet in automated painting) and “workpiece” are used generically. In general, the trajectory of the tool is defined as the Cartesian space path of the tool relative to the workpiece. Trajectory synthesis, in the context of the SCMOs, refers to the process of synthesizing the Cartesian space motion of the tool.

The advent of NC has enabled the manufacturing community to fabricate complex shapes with increasing efficiency. However, to improve the accuracy of the fabricated parts, it is essential to ensure that the coded instructions being sent to the machine tool are themselves correct. The correctness of the coded instructions can be guaranteed—only if the Cartesian space trajectory is designed accurately. This paper seeks to assess the common and most prevalent steps of trajectory synthesis for various SCMOs (for brevity, examples are nonexhaustive and emphasis is placed on the issues in trajectory synthesis), in the context of improving the accuracy and efficiency in fabricating parts.

**Steps in Trajectory Synthesis for Shape Creation**

Any physical object can be viewed as a collection of points in Euclidean space. This is the basis for mathematical models of physical objects based on \( r \)-sets.\(^6\) For physical objects with deterministic boundaries, it was found that the representation of the boundary of the \( r \)-set is equivalent to representing the \( r \)-set itself. Hence, a computer model of a physical object is represented by defining its boundary. Most SCMOs are also based on the fact that a physical object can be represented by its boundary and depend on the ability of a tool to accurately trace the boundary with a series of curves or trajectories. Such SCMOs are referred to as finishing operations as opposed to roughing operations where the tool traces regions inside (or outside) the physical object.

SCMOs can also be classified as curve-based or surface-based SCMOs depending on whether the tool needs to span a curve or surface. Examples of curve-based and surface-based SCMOs are illustrated in Figure 2, where they can be further subdivided as planar or nonplanar SCMOs. For both curve and surface-based SCMOs, the fundamental procedure for trajectory synthesis is common and consists of six broad steps, as follows:

- determining the trajectory layout method
- determining the trajectory generation method
- determining the trajectory spacing method
- determining the tool orientation along trajectory
- postprocessing the trajectory
- verifying the trajectory

These six steps, traditionally based on purely geometric considerations, are reviewed in the context of a relatively new trend of including nongeometric considerations in trajectory synthesis. This new trend is explained below by means of examples.

One example of the new trend is in turning where the deflection of the tool relative to the workpiece changes the effective depth of cut at each point along the trajectory. The deflections of the tool relative to the workpiece, at each point along the trajectory, are modeled to predict a trajectory that would minimize the effects of deflections.\(^6\) Another example is in end milling, where the trajectory is modified from the nominal trajectory (the trajectory based on geometry) to avoid chatter or to compensate for tool deflections.\(^{14,26,71,74,82,84}\) A generalized surface-shaping system that would address the geometric and nongeometric needs in the context of metalcutting was proposed by Hong and Ehmann.\(^{31}\) In rapid prototyping, process parameters along the trajectories and trajectory layouts may be modified to achieve certain material properties of the final part.\(^40\)
The examples presented above can be efficiently and viably implemented only if there exist geometric, material, and process models that interact with each other. Currently, existing geometric models (see Figure 3a) may not be able to provide the information required for physics-based trajectory synthesis due to (a) lack of material information, (b) inability to represent variations from the nominal shape, and (c) the time it takes to interrogate traditional solid models. Research in developing representations for multi-material objects could lead to the development of geometric, material, and process models that interact with each other (see Figure 3b), while providing the ability to tailor such models for a given application.

The rationale for the new trend described above can be explained as follows. Traditionally, the physics of shape-creating operations were taken into account, after trajectory synthesis, by means of servo, process, and supervisory control systems, or by manually manipulating process parameters. The physics was not accounted for until the part was at the machine. Hence, even though the trajectories are optimal with respect to geometric criteria, they may be suboptimal with regard to the entire SCMO. Thus, further demands for increased accuracy and reduced machining time can only be met if the trajectories are synthesized by accounting for both geometric and nongeometric effects of the SCMOs. A disadvantage of the new trend is that the data generation for SCMOs will no longer be machine independent. Another argument against such a trend is that models of manufacturing processes are considered to be complex and highly specialized and, hence, not accounted for except as rules of thumb and tables. These issues could be alleviated by a careful modularization of steps in data generation for SCMOs and would allow users to generate manufacturing data at various levels of detail. Such a modularization will involve manipulating the traditional steps in geometry-based trajectory synthesis.

**Trajectory Layout**

Because one of the requirements of SCMOs is to span planar or nonplanar surfaces with a series of curves (or trajectories), the layout or arrangement of curves on the surfaces has to be specified. The trajectory layout depends on whether the SCMO is curve-based or surface-based.

**Curve-Based, Shape-Creating Operations**

Arc welding and laser machining are two examples of SCMOs where the trajectory of the tool is a set of given curves. Two possible cases arise depending on whether a trajectory layout needs to be specified or not. If the location of the curves (or trajectories) need not be fixed in the workpiece coordinate system (for example, the curves may be translated or rotated), a layout of curves in the domain of the workpiece has to be specified (see Figure 4a). If the location of the curves (or trajectories) in the workpiece coordinate system is fixed, the curves are uniquely defined and no further information is required (see Figure 4b).

In the case of laser machining of sheet metal parts, the objective is to arrange prespecified geometric shapes on a piece of sheet metal by incurring minimum wastage of material. For example, in Figure 4a a set of geometric shapes is redistributed to obtain a better packing density (an extra shape is accommodated). The problem of determining a lay-
out of nonintersecting, closed geometric shapes that reduces wastage of unutilized space is referred to as nesting. While waste of unutilized space is of concern, heat transfer issues will preclude the shapes from being nested too closely, illustrating the importance of nongeometric considerations. In the case of SCMOs, such as arc welding, trajectory layout has no meaning because the trajectory can be uniquely determined.

Surface-Based, Shape-Creating Operations

When the trajectory of the tool is required to span planar or nonplanar surfaces, one needs to determine the layout of trajectories to ensure that the tool spans the entire surface. An example is shown in Figure 5 where a circle (representing a tool) is required to fill a given planar domain. Three common strategies for spanning areas include evolving curves, spiral curves, and space-filling curves (applicable whether the surface is planar or nonplanar).

One criteria in selecting a layout for trajectory synthesis with respect to shape-creating operations is the number of corners or sharp changes in the curvature of the trajectory. Corners or sharp changes in curvature are difficult for NC machines (note that NC machine is not restricted to mean a milling machine) to manipulate (for example, inertia of the machine tool may cause an overshoot) and, hence, need to be avoided. Another criteria in selecting a layout is the necessity to maintain a continuous motion of the tool, without stopping the function of the tool. This criteria is especially important for layouts based on evolving curves where separate curves need to be joined before continuous motion is realized.

Evolving Curves

A trajectory layout strategy that is based on evolving curves involves determining a start curve and traversing the domain with parallel curves. The start curves may be open or closed curves as illustrated in Figure 6. An interesting explanation of the concept of evolving curves can be found in Ref. 68. Commonly encountered layouts that fall under the category of evolving curves are referred to as zigzag, raster, and contour curves. Trajectory layout strategies based on evolving curves are one of the most frequently used layouts in shape-creation operations.

While layouts based on evolving curves are most frequently used in trajectory synthesis for SCMOs, the main drawback is that each curve is a separate entity. Hence, to maintain continuous motion, additional segments need to be created to join the existing curves as shown in Figure 7. Strategies for joining curves that may be applied for most surface-based SCMOs are discussed in detail by Held in the context of pocket machining. Another issue in contour-type layout strategies (especially when offsetting is used) is the need to compute skeletons or medial axes (shown in dashed lines in Figures 6 and 8).
one chooses not to compute the skeletons, the special points (such as boundary points, turning points, singularity points, and self-intersection points) on the offset curve have to be computed.

**Spiral Curves**

Trajectory layouts based on spiral curves present an advantage over strategies based on evolving curves since there is no need to introduce additional joining segments. The most common notion of a spiral is that of the Archimedian spiral, which is the locus of a point moving on a rotating ray, where the point and the ray have uniform, independent motions. This notion may be generalized to layouts that appear to be spirals as shown in Figure 8. Once again, the construction of layouts based on spirals becomes computationally intensive for complex domains because the skeleton or the medial axis (shown as dashed lines in Figure 8) is first required to be estimated. Trajectory layout strategies, based on spirals, rank second in the frequency of usage for SCMOs (after layouts) based on evolving curves.

**Space-Filling Curves**

A space-filling curve is defined as a continuous mapping of a unit line segment onto the unit square. Several types of space-filling curves exist based on the types of initiators (Figure 9a) and generators (Figure 9b) used to create the space-filling curve with any resolution (Figure 9c). The advantage of layouts based on space-filling curves is that there is no need to introduce additional joining segments. In addition, the directionality introduced by evolving and spiral curves is avoided. The disadvantage with space-filling curves is that the number of corners or sharp changes in curvature introduced is large. Another disadvantage is that the trajectory tends to be fragmented into several short lines (or curves). This increases the data being sent to the NC machine. Rounding off sharp corners of space-filling curves and subdividing the surface to optimize the length of the trajectory are examples of techniques employed to overcome the disadvantages of using layouts based on space-filling curves. Examples of the use of space-filling curves as trajectory layout strategies are mostly found in NC milling.

Further research for studying the effects of the directionality of layouts on the geometric properties of the manufactured shape are needed. In addition, research that relates the selected layout to the functional properties of the manufactured shape for various shape-creating operations is needed (along the lines of Kulkarni and Dutta and Li, Dong, and Vickers). Finally, efficient methods for generating layouts (without encountering singularities such as cusps and self-intersections) for geometrically and topologically complex domains are needed.

**Trajectory Generation**

Trajectory generation refers to the process of deriving a mathematical representation of the trajectory in terms of a parametric curve. This step in the trajectory synthesis assumes that there exists a formal computer representation of the desired physical object. The physical object may be represented by a set of curves, a surface model, or a solid model. Note that the representation of the trajectory and curve on the desired physical object are, in general, different.

**Curve-Based, Shape-Creating Operations**

For curve-based SCMOs, one of the first steps involves identifying the curves on the physical object that need to be generated. It is assumed that a computer representation of the curves, appropriately distributed, is available. The process of curve identification may be automated or may involve user intervention. The identification of the curves to be generated is followed by a process of deriving the
mathematical representation of the trajectory. Offset curve calculations and spine curve calculations for rolling ball blends are the two methods commonly employed for deriving the mathematical representation of the trajectory.

An offset of a curve may be described as a curve that is parallel to its progenitor. Offsets of rational curves are, in general, not rational and, hence, need to be approximated. The approximation and properties of offset curves (especially offsets of plane curves) have been studied extensively. Examples of the use of offset curves in curve-based SCMOs can be found in laser machining where the progenitor curve needs to be offset by a distance equal to the radius of the laser beam (see Figure 10a).

Blending becomes relevant in trajectory synthesis, especially for shape-joining operations where the trajectory of the tool is the spine of the blend (Figure 10b). Many mechanical components possess surfaces that meet in tangent discontinuous edges. Such surfaces may be blended for aesthetic purposes or to avoid the effects of stress concentration. A blend (a common notion of a blend is the rolling ball blend) may be visualized as part of the surface created when a spherical ball is rolled over two tangent discontinuous surfaces such that it maintains contact with both surfaces. The spine of a rolling ball blend is the trajectory of the center of the ball. Once again, blending has been studied extensively in the field of computer-aided geometric design.

Surface-Based, Shape-Creating Operations

Trajectory generation for surfaces differs from that for curves and is a two-step process because one only has a representation of the surfaces on the desired physical object. Hence, the first step is to generate representations of curves on the desired physical object. The next step is to generate representations of the trajectory. The focus, therefore, will be on the first step because the second step usually involves taking offsets of curves normal to the surfaces that they lie on.

Planar Surfaces

Trajectory generation for planar surfaces involves calculating the planar curve or line segments that are trajectories of the tool. Calculating the trajectory, in the case of planar surfaces, is totally dependent on the trajectory layout scheme selected. Once the layout has been selected, the common method for trajectory generation involves calculating offset curves as described below.

Offset Curve Calculation. If the geometric elements of the layouts (such as evolving curves, spiral curves, or space-filling curves) are straight lines, the offset curve calculation is trivial because the offset of a line is a parallel line. More important are details such as trimming, blending, and ordering of the offset curves as shown in Figure 11. While Figure 11 shows a single offset curve, the offsetting needs to be carried out until the entire planar domain is covered. If the geometric elements of the layouts are general planar curves, the calculation of offset curves is more involved. Bookkeeping is also important for offsets of general plane curves.

Nonplanar Surfaces

Trajectory generation for nonplanar surfaces is more sophisticated because the trajectories have to be guaranteed to lie on the surface. The methods for
trajectory generation depend on the type of mathematical representation of the surface (that is, parametric or algebraic). Surfaces may also be represented as a cloud of points and normals (or as STL files). The methods for trajectory generation found in the literature for various surface representations are summarized below.

**Isoparameter Curve Calculation.** This method is applicable only when the part surface is represented in the parametric form \( S: r(u, v) \) where \( u \) and \( v \) are the parameters of the surface \( S \). In this method, the trajectory layout is created in the parametric domain of the surface and calculated in the surface domain. Note that any of the layouts mentioned earlier can be implemented in the parametric domain. One of the disadvantages of using an isoparameter curve is that when the part surface is represented as a network of trimmed surfaces, bookkeeping (that is, ordering and maintaining continuity of trajectories) becomes very cumbersome unless each trimmed surface is treated individually.\(^{65}\) The advantage of this method is the ease of calculating isoparametric curves on \( S \).\(^{16,20,48}\)

**Plane/Surface Section Curve Calculation.** This method is applicable for parametric, algebraic, or point-cloud representations of the part surface. A set of parallel planes/surfaces are intersected with the part surface. While any general surface may be used for intersections, planes and cylinders are most commonly used. The advantage of this method lies in the fact that trajectories for networks of surfaces can be generated with ease. Disadvantages of this method are the computational burden involved and the numerical stability of surface intersection algorithms as intersecting surfaces approach tangency.\(^{9,27}\)

**Projection Curve Calculation.** This method is applicable for parametric, algebraic, or point-cloud representations of the part surface. In this method, the trajectory layout is created on a plane. The layout on the plane is then projected, in a given direction, onto the part surface.\(^{12,27}\) The advantage of this method is that it is easy to create layouts in a plane. Disadvantages of this method include the computational burden and numerical instabilities, especially when the layouts to be projected are far away from the part surface.\(^{58}\)

**Offset Curve Calculation.** This method is applicable only when the surface is represented in the parametric or algebraic forms. In this method, offsets of curves on the part surface are considered.\(^{75}\) Since geodesic curves on general surfaces are analogous to straight lines on planes, the offsets are calculated along the geodesic curves normal to the progenitor curve.\(^{49}\) The advantage of this method is that one directly deals with distances along the surface. However, geodesic curves are expensive to compute and offset curves on surfaces suffer from similar problems as those experienced on planes (such as cusps and self-intersections).

**Piecewise Curve Calculation.** Piecewise trajectories are generated by a class of techniques that depend on subdividing the workpiece. Subdivision techniques include cell decomposition, volume decomposition, and artificial intelligence-based methods.\(^{5,69}\) The subdivided domain of the workpiece is used to reason about the cells (or volumes) of material to be kept and cells to be removed. The tool is moved incrementally such that all the cells to be removed are spanned. Such subdivision techniques are usually preferred for roughing operations as opposed to finishing operations.

For the most part, existing CAD software provides the capability to efficiently compute isoparametric, intersection, and projection curves. Research in trajectory generation may provide an impetus for improving computational efficiency in CAD software. Trajectory generation will directly benefit from research efforts in efficient algorithms for computing offsets of curves on surfaces and studies of the properties of such curves.

Novel methods for trajectory generation that include nongeometric issues will provide avenues for improvement. For example, level set methods for propagating a curve normal to itself have shown great potential in image processing. Problems due to cusps and self-intersections, traditionally occurring while computing offset curves, may be avoided by constant or nonconstant distance offsets.\(^{67,68}\) Preliminary results in trajectory generation using nonconstant distance offsets show promise in avoiding cusps and self-intersections.\(^{66}\)

**Trajectory Spacing**

Trajectory spacing refers to the distance between adjacent trajectories in a given layout. Thus, this section is only valid for surface-based SCMOs. Trajectory spacing depends on whether there is exact contact or approximate contact between the tool and desired shape. Exact contact refers to the
capability (at least theoretically) of the tool to exactly generate the desired shape. Examples of exact contact occur in SCMOs such as pocket machining (the desired shape is a plane, and the bottom of the end mill is theoretically capable of generating an exact plane) and robotic spray painting (the desired shape is a surface of given thickness that coats a pre-existing surface). Approximate contact occurs when the tool is capable of exactly generating the desired shape only in the limiting case of spanning every point on the desired shape. Examples of approximate contact occur in milling of free-form surfaces and rapid prototyping.

In both cases, it is necessary to determine the spacing between adjacent trajectories based on minimizing (a) the overall length of the trajectory, and (b) the deviation from desired shape. For example, in fused deposition modeling, point contact between the tool and workpiece causes a deviation from the desired shape. This is illustrated in Figure 12. The spacing between layers may be reduced, thereby reducing the deviation from the desired shape; however, the overall length of the trajectory will increase. In the following sections, trajectory spacing is discussed for the cases of exact and approximate contact between the tool and desired shape.

**Exact Contact**

When there is exact contact between the tool and desired shape, the spacing between the trajectories is relatively straightforward to estimate. The main criteria for estimating the trajectory spacing is that the entire shape should be traversed by the tool without leaving gaps or holes.

One example is pocket machining where there is exact contact between the tool and the desired shape, and the optimal spacing between trajectories is less than or equal to the tool diameter. The criteria for determining the spacing is that no area can be left untouched by the tool. When the spacing between the trajectories is exactly equal to the tool size, triangular regions untouched by the tool are left behind (see Figure 13a). Hence, the trajectory spacing may be reduced or additional segments of the trajectory may have to be included.

Another example is paint spraying where paint is sprayed on the desired shape. Assuming that the spray has a Gaussian layout, the trajectories will be spaced so that the entire shape is covered with paint of the same thickness. This is illustrated in Figure 13b. Once again, note that there exist regions where the paint thickness is uneven.

While determining that the trajectory spacing is relatively straightforward, the problem areas occur (a) at the ends of the trajectories, or (b) when the trajectories have sharp turns where additional segments of the trajectory have to be introduced to remove the excess material left behind (Figure 13a). These cases inevitably increase the length of the trajectories and, therefore, increase manufacturing time. In both of these cases it becomes necessary to take special precautions, such as adding additional segments of the trajectory as in Figure 7. In general, heuristic measures are employed to compromise between the time of manufacturing and the accuracy of the desired shape.
Errors by Swept Volumes and Swept Sections

Approximate Contact
Calculating the trajectory spacing in the presence of approximate contact between the tool and the desired shape necessitates determination of (a) the geometry of the tool relative to the desired shape, and (b) a measure of the error or deviation from the desired shape. Both are critical in accurately estimating the trajectory spacing.

Geometry of the Tool
Approximate contact necessitates the examination of how closely the tool approximates the desired shape. An analysis of SCMOs will show that it is not exactly the geometry of the static tool but the geometry of the moving tool relative to the desired shape that is required. Envelopes or swept volumes have traditionally been used to estimate the geometry of moving objects (tools). Once the envelope has been determined, it can be superposed on the desired shape to get an estimate of the errors. However, calculation of the envelopes or swept volumes are computationally intensive.

An alternative to computing envelopes is computing the two-dimensional approximation of the geometry of the moving tool, referred to as swept sections. The concept of a swept section reduces the complexity of calculations from three dimensions to two dimensions. While the term swept section has been commonly used in milling literature, the concept of swept section has been successfully used for other SCMOs. For example, Figure 12 shows two scenarios of errors assuming two different approximations of the shape of the moving tool (in this case, material deposited by the moving tool).

The accuracy of estimating the deviation from the desired shape, and thus, the trajectory spacing depends on the accuracy of the swept section.

While the geometry of the static tool is fixed, the geometry of the moving tool relative to the desired shape is, in general, constantly changing. A clear understanding of the changing geometry of the tool is necessary for ensuring accuracy of calculations.

Deviation From Desired Shape
Once a suitable representation of the geometry of the moving tool (envelope or swept section) has been derived, a deviation from the desired shape can be estimated. The deviation from the desired shape or the error is defined as the height of the excess (or deficit) material on the manufactured shape measured normal to the desired shape. When the tool is represented by envelopes, the deviation or error may be estimated by Boolean differences between the workpiece and enveloped volumes. When the tool is represented by swept sections, pointwise error estimates need to be calculated. Both cases are demonstrated in Figure 14. The tool envelopes or tool-swept sections may represent portions of the tool that are in the vicinity of the desired shape and need not represent the entire tool unless it is required (this reduces the computational burden). The trajectory spacing is determined so that the deviation from the desired shape (the error) does not exceed user-specified limits.

In general, trajectory spacing involves iterative numerical calculations. To avoid iterative calculations, the most commonly employed measure is to approximate the desired shape in the plane of the swept section. The desired shape is approximated by the circle of curvature at the point under consideration as illustrated in Figure 15, where the tool trajectories are assumed to be locally perpendicular to the plane of the swept sections at P and Q. The trajectory spacing is calculated based on the assumption that both swept sections P and Q lie in the plane of the circle of curvature (this is not true except for very simple geometries). In addition, depending on the curvature at the point P, the actual trajectory spacing (see Figure 15a) can be quite different from the calculated trajectory spacing (see Figure 15b).

Trajectories can be spaced based on (a) constant distance between trajectories, (b) maximum allowable error relative to the desired shape, or (c) con-
stant error relative to the desired shape. In (a) the distance between trajectories is maintained at a constant value either in the parameter or object domains. In (b), the error relative to the desired shape, between the trajectories, is kept below a certain user-specified value. In (c), a constant error relative to the desired shape is maintained between trajectories. Any type of trajectory generation method may be used if the trajectory spacing is based on (a). Isoparametric curve generation, planar curve generation, and projection curve generation methods only allow the user to specify a maximum allowable error (this results in unwanted clustering of trajectories in these cases), while the offset curve generation method will allow the user to specify both the maximum allowable error or constant error. Numerical experiments have demonstrated that the overall length of the trajectories decreases in the order (a), (b), and (c).\textsuperscript{32.75}

As in the exact contact of trajectory spacing, regions of sharp turns in the tool trajectory and ends of the tool trajectory are causes for concern as more/less material remains in those regions than otherwise estimated.

Traditionally, swept sections of tools have been modeled as simple geometric shapes (such as rectangles, ellipses, and so on) for ease of calculations. Such models are valid under the assumption that the tool moves locally along a straight line and does not change orientation. However, as the tool motion becomes complex, as it does in five-axis machines, the swept sections may deviate from the traditional geometric shapes. Studies that evaluate the accuracy of using swept sections and procedures that accurately calculate swept sections are necessary. In addition, techniques that include the effects of non-geometric errors when calculating the trajectory spacing are required.

**Tool Orientation**

For NC machines that have more than three axes, the inclination of the tool along the trajectory needs to be specified. While machines with multiple (more than three) axes have the advantage of enhancing the accessibility of the desired shape, an additional step in determining the tool orientations is introduced. Tool orientations are currently estimated based on avoiding local and global accessibility problems of the tool. Note that for NC machines with three or lesser axes, local and global accessibility problems are avoided by restricting the range of desired shapes and tools. Because tool orientations are typically determined after the trajectory generation, this section applies equally well for curve-based and surface-based SCMOs that operate with multiple axes NC machines.

**Local Accessibility**

Local accessibility problems occur when the tool removes/adds excess material as it comes into contact with the desired shape. The detection of local accessibility problems is closely dependent on the specific shape-creation operation under consideration. Methods for detecting problems in local accessibility are based on studying the geometry of the tool relative to the local region of contact between the tool and the desired shape. The criteria for detecting local accessibility problems can be generalized by the following steps: (a) determine the radius of curvature $r_1$ of the desired shape in a plane $(P)$ locally perpendicular to the direction of tool motion, (b) determine the radius of curvature $r_2$ of the swept section of the tool in the plane $P$, and (c) if $r_2 < r_1$, then local accessibility problems do not occur at the point under consideration, otherwise local accessibility is an issue and precautions need to be taken such as reducing the radius of the tool being used and changing the inclination of the tool so that its effective radius of curvature decreases.

Local accessibility issues have received much attention in milling where it is critical that the tool does not remove excess material.\textsuperscript{34,44,45,51} Local accessibility is not much of an issue in noncontact shape creation operations such as laser machining and selective laser sintering.
Global Accessibility

Problems in global accessibility occur when the tool comes into contact with the machine, fixtures, or parts of the desired shape that were not intended to be machined. Global accessibility can be detected by using the concept of accessibility cones. Accessibility cones are a set of feasible directions that ensure a particular point is reachable with the tool. Finding accessibility cones is quite computationally intensive.

Feasible Tool Orientations

Feasible tool orientations are those that satisfy both the local and global accessibility criteria. In general, composite accessibility cones (that is, those directions that satisfy both local and global accessibility criteria) are used to obtain a set of feasible tool orientations at each point along the toolpath. Tool directions are picked from the accessibility cones based on maximizing local material removal/addition and interpolated to obtain a smooth tool motion. Local and global smooth motion interpolation will result in better kinematic characteristics of the toolpath.

Additional work needs to be done in the optimal selection of key positions and orientations that allow smooth motion interpolation in the presence of kinematic and dynamic considerations. Existing work on interpolating orientations and generating quaternion splines may prove useful.

Trajectory Interpolation

NC machines are characterized by their basic length unit (BLU) and sampling period. Both of these quantities dictate the accuracy of the traversed trajectories. However, improperly sampled trajectories give rise to contour errors and feedrate errors even though the NC machine may have a small BLU and sampling period. Thus, the process of sampling the curve that represents the trajectory is very important in ensuring the accuracy of the traversed trajectories. While sampling of curves is also studied in computer graphics where time is not an issue, the challenge for NC machines lies in the multi-axes coordinated motion. In NC terminology, the process of sampling is referred to as interpolation, and the hardware/software that does the interpolation is referred to as an interpolator.

Several interpolators (both on-line and off-line in hardware and software) have been proposed for interpolating parametric curves based on splitting parametric curves into lines and arcs. Most machine controllers are capable of tracking line segments and circular arcs. This requires that the trajectory be split into arc/line segments. The fact that trajectories have to be split has an affect on the performance of the machine as follows:

1. Splitting of curves into line and arc segments causes discontinuities in the derivatives along the trajectory.
2. To maintain the smoothness of the trajectory, it is necessary to sample a very large number of points along the trajectory.
3. The memory requirements of NC machines increase due to the need to store larger amounts of data.
4. The communication load between the NC machine and CAM system increases.
5. Shorter segments do not allow time for the feedrate to reach its specified value. Therefore, fluctuations in the feedrate occur.

More recently, spline interpolators that can accurately track general curves have become popular. While research in trajectory interpolation has been mainly focused on milling machines, it is equally applicable for other shape-creating operations that use NC machines.

Research has been performed to interpolate general curves. This research covers three-axis machines; however, further research in interpolation for machines with more than three axes or in interpolating other process parameters (especially in the presence of nongeometric influences) will be beneficial to improve the accuracy of SCMOs.

Trajectory Verification

Trajectory verification is a posteriori check to ensure that the trajectory is valid (that is, locally and globally accessible). Several methods exist for verifying the trajectories (the methods are related to milling but applicable for any shape-creation operation) as outlined below.

Boolean Difference Modeling. The volume swept by the tool as it moves along the trajectory is referred to as the swept volume. Verification is achieved by performing Boolean differences between models of the latest workpiece (which is the difference between the previous workpiece and
swept volumes) and desired shape. One of the difficulties of this approach is the computational burden incurred in computing swept volumes and Boolean differences.

**Z-Buffer Modeling.** The Z-buffer algorithm is a standard algorithm used in computer graphics for hidden-line elimination. A similar algorithm is adapted for trajectory verification. A vector normal to the plane of the screen, referred to as a scan line, is drawn at each pixel. For each pixel, the Z heights of the intersecting points on the desired shape and workpiece are stored. The swept volume of the tool is replaced by a polyhedral approximation. The scan line is intersected with the polyhedral approximation and the Z heights computed. The workpiece Z buffer is updated by comparing with the swept-volume Z buffer. After completion, the Z buffer of the workpiece is compared to the Z buffer of the desired shape. The disadvantage of this method is that it is view dependent.

**Discrete Modeling.** This approach consists of three steps. The first step involves discretizing the desired shape into appropriately distributed points and normals. Next, discretized points that are close to the envelope of the moving tool are identified. This is done by tracking the distances from the points, along the normals, to the envelope of the moving tool in a manner similar to the Z-buffer algorithm. Finally, the normals are intersected with representations of the swept volume to get accurate values of the distances.

**Voxel Modeling.** Voxels are volumetric elements in which a solid may be partitioned. Voxels have been used to obtain volumetric representations of the desired shape, workpiece, and tool. Instead of dealing with the exact representations of the desired shape and tool-swept volume, voxel-based approximate representations are used to check for intersections of the tool and workpiece. Heuristic methods are used to determine a subset of voxels that have the greatest potential of lying in the intersecting regions. Those voxels, which are in regions where the tool intersects with the workpiece, are tagged as voxels to be removed. If the tagged voxels are part of the desired shape or part of the fixtures, the user is warned.

**Ray Casting.** Ray casting is a means of representing a physical object by a set of parallel line segments that are classified as lying inside or outside the physical object. Using ray representations of physical objects, reduced dimensional Boolean operations are possible. Ray casting has also been proposed to track changes in workpiece geometry, machine dynamics, cutting forces, and so on. Trajectory verification will continue to be an active area of research where solutions are tailored for different applications to reduce the computational burden.

**Summary**

Issues in trajectory synthesis for SCMOs have been assessed in the context of a new trend of including nongeometric criteria during trajectory synthesis. Currently, the trajectory layout, generation, spacing, postprocessing, and verification are primarily based on geometric calculations. Each of the steps in trajectory synthesis will have to be enhanced to include nongeometric criteria.

**Acknowledgments**

Partial support from the National Science Foundation grant DMI-9713818, an Iowa State University research grant, and the Society of Manufacturing Engineers Research Initiation Grant 597-2401 is gratefully acknowledged. The author would like to acknowledge Vinod Kumar and Prashant Kulkarni from the University of Michigan and Dr. Pal Molian from Iowa State University for reading early drafts and providing many suggestions and comments.

This paper first appeared in the *Proc. of the 1998 ASME Design Engineering Technical Conference* and was submitted to the *Journal of Manufacturing Systems* with permission from the American Society of Mechanical Engineers (ASME), New York.

**References**

8. J.E. Bobrow, "Solid Modelers Improve NC Machine Tool Path

Author’s Biography

Radha Sarma is a faculty member in the Dept. of Mechanical, Industrial, and Manufacturing Engineering at the University of Toledo, OH. A previous three-year appointment was in the Dept. of Mechanical Engineering at Iowa State University. Sarma obtained her master's degree from the University of Toledo and PhD from the University of Michigan. Her research interests include solid modeling, surface modeling, CAD for MEMS, and CAM.