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Origins of Numerical Knowledge

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Abstract

Evidence is presented that young human infants possess a system of numerical knowledge that consists of a mechanism for determining and representing small numbers of entities, as well as procedures for operating over these representations so as to extract information of the numerical relationships between them. A model for this mechanism is presented, and its relation to the development of further numerical knowledge is discussed.

Introduction

In this paper, I will make three central arguments. First, I will argue that human infants possess extensive numerical competence. Empirical findings show that young infants are able to represent and reason about numbers of things. Infants' ability to determine number is not based on perceptual properties of displays of different numbers of items, nor is it restricted to specific kinds of entities such as physical objects. Rather, it spans a range of ontologically different kinds of entities. Infants therefore have a means of determining and representing number *per se*. Furthermore, infants are able to reason numerically about these representations; they possess procedures for operating over these representations in numerically meaningful ways, and so can appreciate the numerical relationships that hold between different numerical quantities. They can thus be said to possess a genuine system of numerical knowledge. These early capacities suggest the existence of an unlearned basic core of numerical competence.

Second, I will propose a mechanism that underlies this early numerical competence, the Accumulator Mechanism, and discuss the structure of the numerical representations that it produces. In particular, I will argue that our initial representations of number have a magnitudinal structure in which the numerical relationships that hold between the numbers are inherently specified.

Finally, I will address the issue of how the numerical competence arising from the Accumulator Mechanism relates to more advanced numerical and mathematical knowledge. There are limits to the mathematical knowledge the proposed mechanism could give rise to, and evidence, both developmental and historical, of humans' acquisition of aspects of mathematical knowledge beyond these limits lends further support for this mechanism. Furthermore, there are significant differences in the representations of number provided by the Accumulator Mechanism, and those instantiated in linguistic counting. Thus, despite possession of an early appreciation of number, learning the system of the counting routine and learning the meanings of the number words are difficult and complex achievements for children.

Numerical Capacities of Infants

Infants' Sensitivity to Numerosity

"Subitization" is the term for the ability to automatically "recognize" small numbers of items without having to engage in conscious counting. In typical experiments, adult subjects are presented briefly with displays containing items such as black dots, and asked to name as quickly as possible the number of items displayed. Subitization is characterized by a very shallow reaction time slope, and by near-perfect performance, in the range of 1 to 3 or 4 items. Beyond this, the RT slope typically increases steeply, and errors become more frequent, indicating a much slower, non-automatic mental process applies to enumerating items in displays outside of this range (e.g., Chi & Klahr 1975; Kaufman, Lord, Reese, & Volkmann 1949; Mandler & Shebo 1982).

Findings suggest that the ability to subitize is inborn. It has been reliably demonstrated that human infants can distinguish between different numbers of items. When infants are repeatedly presented with displays containing a certain number of items until their looking time to the displays drops off (indicating a decrease in interest), they will then look longer at new displays (evidencing renewed interest) containing a new number of items (e.g., Antell & Keating 1983; Starkey & Cooper 1980; Strauss & Curtis 1981; van Loosbroek & Smitsman 1990). This kind of habituation paradigm has found that infants can distinguish 2 from 3, and, in certain conditions, 3 from 4 (e.g., Strauss & Curtis 1981, van Loosbroek & Smitsman 1990) and even 4 from 5 (van Loosbroek & Smitsman 1990).

Given that the upper limits of numerical discrimination in infants correspond to the limits of the subitizable number range in adults, it seems likely that the same quantification process underlies both capacities. This suggests that a better understanding of the subitization process can inform our understanding of the nature of infants' early numerical concepts; and, conversely, that improved knowledge of infants' numerical discrimination abilities may help shed light on the nature of subitization.

Different proposals have been made as to the nature of subitization. One view is that at root, it is not a process that actually determines numerosity; rather, it is a pattern-recognition process that holistically identifies perceptual patterns that correlate with arrays containing a certain number of items (e.g., Mandler & Shebo, 1982). On this view, children must somehow 'bootstrap' themselves up into an ability to represent numbers of things, from an initial ability to recognize and represent the perceptual patterns that piggyback along with small-sized collections.

However, there is now ample evidence that subitization is not a pattern-recognition process. For one thing, habituation studies have varied the perceptual attributes of the items in the displays presented to infants. In one experiment, infants were presented with a slide picture of either 2 or 3 household objects arranged in random locations (Starkey, Spelke & Gelman 1990). The objects within and between trials were all different -- no object was presented more than once. Objects were of different sizes, colors, and textures, and differed in complexity of contour (e.g., an orange, a glove, a keychain, a sponge, sunglasses, etc.). Following habituation, infants were alternately presented with new pictures of 2 and of 3 such objects. Despite the lack of perceptual similarity, infants successfully discriminated pictures on the basis of their number, looking longer at pictures showing the novel number of objects.

Furthermore, infants' numerical discrimination is not restricted to visual arrays. Infants can also enumerate sounds, and can recognize the numerical equivalence between arrays of objects and sounds of the same number (Starkey, Spelke & Gelman 1990). In one experiment, infants were habituated to slide pictures of either 2 or 3 objects like those described above. Following habituation, infants were presented with a black disk that emitted alternately 2 drumbeats, and 3 drumbeats. Infants looked longer at the disk when it emitted the same number of drumbeats as the number of items they had been habituated to visually, indicating that (a) infants were enumerating the drumbeats, and (b) the numerical representations resulting from their enumeration of sounds and of objects is a common one, in that it can be matched across these two ontologically distinct kinds of entities.

Experiments recently conducted in my infant cognition laboratory indicate that infants' enumeration capacities apply even more broadly. We have been investigating 6-month-olds' ability to enumerate another ontologically different kind of entity: physical actions (Wynn, 1995). A sequence of actions such as those used in these experiments -- the repeated jumps of a single puppet -- differs from displays of objects typically presented to infants in several respects. Objects in an array have an enduring existence, exist together in time, and occupy distinct portions of 2-dimensional (in the case of photographs or pictures) or 3-dimensional space. Actions in a sequence, on the other hand, have a momentary existence, occupy distinct portions of time, and may or may not occur at precisely the same location in space. Perhaps most importantly, perceptual access is available to the entire collection of objects at once, but to just one element at a time of a collection of sequential actions, so that one cannot anticipate the final element. Actions also differ from sounds in several ways. There are specific material objects or "agents" associated with actions, whereas sounds are disembodied entities -- though a sound may emanate from a specific physical object, it is perceived independently of the object in a way that an action cannot be. Because of this, sounds are (psychologically, if not physically) inherently temporal entities, whereas actions are inherently spatiotemporal; both spatial and temporal information together specify an action, making the identification of an action a complex task requiring the integration of spatial information over time.

Furthermore, actions are interesting to study because Frege's (1893/1980) point that number is not an inherent property of portions of the world applies every bit as much to actions as it does to physical matter. Just as a given physical aggregate may accurately be described as *52 cards*, *one deck*, or *10^{28} molecules*, if I walk across a room it counts as one "crossing-the-room" action, but may count as five "taking-a-step" actions. Number of actions is only determinate relative to a sortal term that identifies a specific kind of action. There is no inherent, objective fact of the matter as to where, in the continuously evolving scene, one 'action' ends and the next 'action' begins. The individuation of discrete actions from this continuous scene is a cognitive imposition.

Six-month-old infants were presented with a display stage containing a puppet. Half the infants were habituated to a puppet jumping 2 times, the other half, to a puppet jumping 3 times. During the habituation phase of the first experiment, on each trial the puppet jumped the required number of times, pausing briefly between the jumps. Upon completion of the jump sequence, the infant's looking time to the now-stationary puppet was recorded. In the test phase of the experiment, infants were presented with trials in which the puppet alternately jumped 2 times and 3 times. The structure of the jump sequences ruled out the possibility that infants could be responding on the basis of tempo of jumps, or overall duration of jump sequences. The test sequences containing the old number of jumps always had both a different tempo and a different overall duration from the habituated sequence. The test sequences containing the novel number of jumps differed from the habituated sequence on only one of these dimensions (for half the infants tempo, for half, duration), having the same value on this dimension as the test sequence with the old number of jumps. Therefore, with respect to duration and tempo, the test sequence with the old number of jumps was the most different from the habituated sequence.

Infants looked significantly longer at the puppet after it jumped the new number of times than after it jumped the habituated number of times (see Figure 1). To exhibit this pattern of preferences, infants must have been sensitive to the number of jumps in each sequence.

INSERT FIGURE 1 ABOUT HERE

In order to begin investigating what cues infants use to individuate the events -- how infants decide where one action ends and the next begins -- a second experiment was conducted, exploring the possibility that infants were segmenting the sequence on the basis of the presence and absence of motion. In the first experiment, infants may have been detecting and enumerating those periods of time in which the puppet was in motion, which were bounded in time by periods in which the puppet was motionless. In the next experiment, therefore, we presented infants with sequences of jumps in which the puppet was in constant motion -- between jumps, the puppet wagged from side to side. As in the first experiment, one group of 6-moth-olds was habituated to sequences of 2 jumps, another, to sequences of 3 jumps. Following habituation, infants were presented with trials in which the puppet alternately jumped 2 and 3 times.

INSERT FIGURE 2 ABOUT HERE

Again, infants were sensitive to the number of jumps, looking significantly longer at the puppet following the new number of jumps (see Figure 2). Thus, infants are able to parse a sequence of continuous motion into discrete segments, and are able to enumerate the individuals so arrived at.

These findings have two implications. First, they show that infants have some concept of 'individual' that can apply to physical action; they possess a cognitive process for imposing some as-yet-undetermined criteria for individual-hood over continuous spatial and motion input. Whether this notion of 'individual', and the accompanying principles of individuation that specify such individuals, are the same as those which specify individual physical objects, individual sounds, and possibly other kinds of individuals as well, remains to be determined (see Bloom, 1990, 1994b for evidence that a single notion underlies the conception of 'individual' across different domains).

Second, the fact that infants can enumerate different ontological kinds of entities ranging from objects, to sounds, to physical actions, indicates that the mechanism for determining number is quite general in the kinds of entities it will take as input to be counted. It may even take in *anything* the cognitive/conceptual system specifies as an individual. This generality, in turn, indicates that the enumeration mechanism cannot be operating over physical or perceptual properties of the entities, recognizing certain perceptual patterns, and the like; it must be determining number of entities per se.

Do Infants Possess a System of Numerical Knowledge?

The studies reviewed above show that human infants are sensitive to number itself; they have representations of number that apply over a wide range of perceptually different situations and different kinds of entities.

But possessing genuine numerical knowledge entails more than simply the ability to represent different numbers. A numerical system is composed not solely of numbers, but also of procedures for the manipulation of these numbers to yield numerical information; for example, information of how two numerical values relate to each other. Infants may be able to determine numbers of things without being able to reason about these numbers or to make numerical kinds of inferences on their basis. If so, we would not want to say that infants' representations of these small numbers play numerically meaningful roles in the infant's cognitive system, nor to credit infants with an actual numerical understanding, with a system of numerical knowledge.

Cooper (1984), for example, has proposed that human infants' initial ability to discriminate small numbers of items does not include any ability to relate these numbers to each other; infants' concepts of, say, 'oneness', 'twoness' and 'threeness' are initially as unrelated to each other as are our adult concepts of, say, 'clocks' and 'chickens'. That is, they have no superordinate concept of *number* (or of *numbers of things*) to unite their concepts of individual numbers. On this view, infants gradually come to learn that the numbers belong to a single category by repeatedly observing situations where 1 or more objects are added to or removed from small collections of objects. From such observations, infants see that the actions of addition and removal reliably result in a change from one number to another, and learn the orderings of the numbers and the relationships that hold between them. Thus they build up a superordinate concept, *number*. Quite similar views have also been expressed in the philosophy of mathematical epistemology; Kitcher (1984), for example, has proposed that "Mathematical knowledge arises from rudimentary knowledge acquired by perception", and that "Children come to... accept basic truths of arithmetic by engaging in activities of collecting and segregating."

Studies in my laboratory show that, in contrast to these kinds of learning accounts, 5-month-old infants are able to engage in numerical reasoning; they possess an appreciation of the numerical relationships that hold between different small numbers. Thus, they have procedures for manipulating these numerical concepts to attain numerical information.

In these experiments, infants are shown a small collection of objects which then has an object added to or removed from it. The resulting number of objects shown to infants is either numerically consistent with the events, or inconsistent. Since infants look longer at outcomes that violate their expectations, if they are anticipating the number of objects that should result, they will look longer at the inconsistent outcomes than the consistent ones.

In the first experiments (Experiments 1 and 2 of Wynn, 1992a), 5-month-old infants were divided into 2 groups. One group was shown an addition situation in which 1 object was added to another identical object, while the other group was shown a subtraction situation in which 1 object was removed from a collection of 2 objects (see Figure 3). Infants in the "1+1" group saw one item placed into a display case. A screen then rotated up to occlude the item, and the experimenter brought a second item into the display and placed it out of sight behind the screen. The "2-1" group saw 2 items placed into the display; the screen rotated up to hide the 2 items; then the experimenter's hand re-entered the display, went behind the screen, and removed 1 item from the display. For both groups, the screen then dropped to reveal either 1 or 2 items. Infants' looking time to the display was then recorded. Infants received six such trials, in which the consistent and inconsistent outcomes were alternately revealed.

INSERT FIGURE 3 ABOUT HERE

Pretest trials, in which infants were simply presented with displays of 1 and 2 items to look at, revealed no significant preference for one number over the other, and no significant difference in preference between the two groups. But there was a significant difference in the looking patterns of the two groups on the test trials; infants in the 1+1 group looked longer at the 1-item result than at the 2-item result, while infants in the 2-1 situation looked longer at the result of 2 items than the result of 1 item (see Figure 4).

INSERT FIGURE 4 ABOUT HERE

These results show that 5-month-olds know there should be a change in the display as a result of the operation. In both situations, infants looked longer when the screen dropped to reveal the same number of objects as there were before the addition or subtraction. But these results do not show they are anticipating the precise nature of the change. They may simply be expecting the display to be changed in some way from its initial appearance, or expecting there to be a different number of objects without expectations of a *specific* number. To address this issue, another experiment was conducted (Experiment 3 of Wynn, 1992a). Five-month-olds were shown an addition of one item to another, in which the final number of objects revealed was either 2 or 3. Since both outcomes are different from the initial display of one object, then if infants do not have precise expectations of how the display should be changed, they should be equally unsurprised by both outcomes and so look equally. However, if infants are computing the exact result of the addition, they will expect 2 items to result, and so should look longer at the result of 3 items than of 2 items.

This was the pattern observed; infants looked significantly longer at the inconsistent outcome of 3 objects than at the consistent outcome of 2 objects (see Figure 5). (Pretest trials revealed no baseline preference to look at 3 items over 2 items.) These results show that infants are computing the precise result of the operation; they know not only that the display of objects should change, but exactly what the final outcome should be.

INSERT FIGURE 5 ABOUT HERE

The findings obtained from these studies are quite robust; they have been obtained in a number of other laboratories, using different stimuli and with variations in the procedure. Baillargeon, Miller and Constantino (1994; see Baillargeon 1994) showed 10-month-olds an addition situation in which a hand deposited first one, and then another, item out of sight behind a screen. The screen was then lowered, revealing either two or three items. The infants looked longer when shown three items behind the screen than when shown two, showing that they had been expecting only two. (They had previously demonstrated equal preference to look at two versus three items.) Similarly, Uller, Carey, Huntley-Fenner, and Klatt (1994) obtained longer looking at the numerically incorrect result when they presented 8-month-olds with "1+1" situations that resulted in either 1 or 2 items.

In another study, Moore (1995) presented one group of 5-month-olds with "1+1" situations, and another group with "2-1" situations; for both groups the outcome alternated between 1 and 2 items. All sequences of events were presented on a computer display. The "objects" were computer-generated rectangular random checkerboards, and the "screen" was a red square which descended from the top of the computer display to obscure the checkerboard(s), and ascended to reveal the "outcomes". Despite the shift from actual, 3-dimensional objects to 2-dimensional, non-representational computer-generated patterns, the effect was still obtained; each group of infants looked longer at the impossible outcome.

In another variation of this paradigm, Simon, Hespos & Rochat (in press) obtained results that rule out the possibility that infants might be computing over visual aspects of the display, expecting an outcome scenario containing particular visual elements, rather than expecting a specific number of objects. They presented one group of 5-month-olds with "1+1" situations, and another group with "2-1" situations; for both groups the outcome alternated between 1 object and 2 objects. But in addition to manipulating numerical possibility, they manipulated "identity possibility" -- on some trials, the items in the outcome display were the kind of object that should be expected, while in other trials, one of the items in the outcome display had an unexpected identity. For example, in a situation in which 1 Ernie doll was added to another Ernie doll, the outcome display might be: 2 Ernie dolls (numerically and identity correct), 1 Ernie doll and 1 Elmo doll (numerically correct but identity incorrect), 1 Ernie doll (numerically incorrect but identity correct), or 1 Elmo doll (numerically and identity incorrect). Infants looked longer at the outcomes that were numerically incorrect, regardless of the identity of the dolls (despite a control condition showing that infants could perceptually distinguish Elmo from Ernie). Therefore, they could not have been responding on the basis of expectations about the precise visual aspects of the display.

In a particularly elegant study, Koechlin (1994) tested the possibility that infants were anticipating certain spatial locations to be filled and others empty, rather than anticipating number. In the original Wynn (1992a) studies, infants may have been able to determine the approximate location behind the screen where the second object was placed (in the "1+1" situations) or removed from (in the "2-1" situations). In the impossible outcomes, therefore, they may have been surprised not because there was an incorrect number of objects, but because a location was empty when it should be filled ("1+1" impossible outcome), or filled when it should be empty ("2-1" impossible outcome). In this experiment, 5-month-olds were presented with "1+1" and "2-1" situations as in Wynn (1992a). However, all objects were placed on a large revolving plate that was located behind the screen, so no object retained a distinct spatial location throughout the experimental operation. Nonetheless, infants looked reliably longer at the numerically incorrect outcomes, showing that they were computing over number of objects, not over the filled/empty status of different spatial locations.

Finally, an intriguing extension of this paradigm has obtained similar results in non-human primates (Hauser, MacNeilage and Ware, 1995). In these studies, rhesus monkeys were presented with the following situation: one eggplant was placed in an open box where the monkey could see it; a partition was then placed in front of the eggplant obscuring it from view; the monkey then saw a second eggplant placed in the box; the partition was then removed to reveal either 1 or 2 eggplants in the box. Although the monkeys showed no preference in control situations to look at 1 eggplant over 2, in this situation they looked significantly longer at the 1-eggplant outcome, indicating they had been expecting 2 eggplants to be in the box.

We have recently extended these results to further numerical situations, exploring whether infants have expectations about the results of situations in which 1 item is added to 2 items, or 1 is taken away from 3 (Wynn, in preparation). In this experiment, we modified the methodology somewhat. Previously we had shown infants the same sequence of events on all trials, alternately resulting in 2 different outcomes. Here, we presented infants with exactly the same outcome on all trials, but with 2 different sequences of events that led up to the outcome -- on half the trials, the events were consistent with the outcome, on the other half, they were inconsistent.[1] The motivation for this modification was to remove any influence on test looking times of possible subtle preferences for looking at one number of objects more than another.

Infants were alternately presented with "2+1" and "3-1" trials. In the "2+1" trials, infants saw 2 items initially in the display, the screen rotated up to hide them, and then a hand entered the display with a 3rd item, placed it behind the screen along with the other two, and left the display empty. In the "3-1" trials, the display initially contained 3 items, the screen rotated up to hide them, and then an empty hand entered the display and removed one of the objects. For half of the infants, the result of these two sequences was always 2 items -- the correct outcome for the 3-1 sequence, but incorrect for the 2+1 sequence. For the other infants, the result was always 3 items -- the correct result for the 2+1 sequence but not the 3-1 sequence.

Infants in the two groups should give different patterns of looking if they are anticipating the outcomes of the events. Infants for whom the presented outcome is always 2 objects should look longer on the "2+1" than on the "3-1" trials if they have expectations about what the result of 1 object added to 2 should be. However, this result would not suggest that they appreciate that 1 object removed from 3 results in 2 objects, since they might be looking less long on 3-1 trials *either* because they appreciate that the result is numerically correct, *or* because they have no clear expectations about the result in this case. Infants for whom the outcome is always 3

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[1]I thank Renee Baillargeon for suggesting this methodological modification to me.

objects, on the other hand, should look longer on the "3-1" than on the "2+1" trials, if they have expectations about what the outcome of 1 object removed from 3 should be. Again, however, this result would tell us only about their expectations in the "3-1" situation; a lack of surprise on the "2+1" trials could be either because the outcome of 3 matches their expectation, or because they have no clear expectations about the result in this case.

Infants in the two groups differed significantly in their looking patterns, in the predicted direction. Infants for whom the outcome was 2 looked longer following the "2+1" than the "3-1" situations, while those for whom the outcome was 3 showed the reverse pattern (see Figure 6). However, separate analyses of each group's looking times revealed that only the 3-outcome infants significantly distinguished between the consistent and inconsistent events; though the 2-item infants looked longer following the sequence of events inconsistent with the outcome, this difference was not significant. In this experiment, therefore, infants showed an appreciation of the outcome of three objects with one removed, but not of the outcome of two objects with one added. Inferring the results of the addition in these experiments appears to be more difficult for some reason than inferring the results of the subtraction.

INSERT FIGURE 6 ABOUT HERE

The many experiments reviewed above show that 5-month-olds appear to be sensitive to the numerical relationships between small numbers of objects. This implies that (a) infants' representations for these numbers possess a structure that embodies this kind of numerical relational information, and (b) infants possess a means of operating over their representations of these numbers to extract this information. I will now present a theory of an innate mechanism for determining number that entails such a structure to the numerical representations it yields, and is amenable to straightforward processes by which information of the numerical relationships between these representations could be obtained.

A Mechanism for Determining and Reasoning About Number

This model, the "Accumulator Mechanism", was originally proposed to account for rats' ability to discriminate number (Meck & Church, 1983). It has recently been extended to account for human infants' preverbal numerical competence and for nonverbal numerical processes such as subitization (Gallistel & Gelman, 1991, 1992; Wynn, 1990a, 1992c).

Meck and Church (1983) suggest that a single mechanism underlies both animals' ability to determine number, and their ability to measure duration. The model operates as follows: a pacemaker puts out pulses at a constant rate, which can be passed into an accumulator by the closing of a switch. When this mechanism is in its counting mode, every time an entity is experienced that is to be counted, the switch closes for a fixed brief interval, passing the pulses into the accumulator during that interval. Thus the accumulator fills up in equal increments, one for each entity counted. In its timing mode, the switch closes at the beginning of the event to be timed, and remains closed for the duration of the interval, passing energy into the accumulator continuously. The final fullness level of the accumulator represents the number of items counted or how much time has gone by. The mechanism contains several accumulators and switches, so that the animal can count different sets of entities and measure several durations simultaneously. The fullness value in the accumulator can be passed into working memory and compared with previously stored accumulator values, to allow the animal to evaluate whether a number of items or events (or a duration) is greater, smaller, or the same as a previously stored number (or duration).

Note that there will be differences in the exact fullness of the accumulator on different counts of the same number of items, resulting from inherent variability in the rate at which the pacemaker is generating pulses and in the amount of time the switch closes for each increment. This variability is normally distributed around a mean fullness value for each number, and obeys Weber's law; the variance is greater for larger numbers than it is for smaller ones and increases in proportion to the numerosity represented. Thus, as number increases, the discriminability of adjacent numbers decreases (see Gallistel & Gelman, 1991 for discussion). This accords with data showing decreasing discriminability with increasing numerosity in animals (e.g., Mechner & Guevrekian, 1962; Platt & Johnson 1971), in human adults (e.g., Chi & Klahr, 1975; Mandler & Shebo, 1982), and in human infants (e.g., Starkey & Cooper 1980; von Loosbroek & Smitsman 1990; Wynn, 1995).

Meck & Church (1983) obtained experimental support for the claim that the same mechanism underlies animals' counting processes and their timing processes. First, the administration of methamphetamine increases rats' measure of duration and of number by precisely the same factor, suggesting that it is a single mechanism being affected. This effect can be explained in the model by positing that the drug causes an increase in the rate of pulse generation by the pacemaker, leading to a proportionate increase in the final value of the accumulator regardless of its mode of operation. Second, they tested the following prediction: If a rat's decision (e.g., when or whether to press a lever for food reward based on presented stimuli) is based on comparing the final value of the accumulator with a previously stored value, then it might not matter whether the previously stored value is the result of a timing process or a counting one. For example, a count that results in the same final fullness value in the accumulator as a previously trained duration might be responded to as if it were that duration. This prediction was confirmed -- when rats were trained to respond to a specific *duration* of continuous sound, they immediately generalized their response when presented with a certain *number* of 1-second sound segments that had been calculated by the experimenters to fill up the accumulator to the same level as that for the duration the rats had been initially trained on. Similar transfer results were obtained when rats were initially trained to respond to a specific number of events, and then tested on a duration of continuous sound. Meck and Church concluded that it was indeed the same mechanism underlying both counting and timing processes in rats.

Because it is the *entire fullness of the accumulator* that represents the number of the items counted, the magnitudinal relationships between the numbers are inherently specified in the accumulator's representations for them. For example, the number 6 is 3 more than the number 3, or twice as large; and the accumulator's representation for 6 has three more increments than the representation for 3, and so the accumulator is twice as full. Provided the infant or animal has certain procedures for operating over these representations, it will be able to extract some of these numerical relationships. Addition, for example, could be achieved by combining or concatenating two numerical representations; "pouring" the contents from an accumulator representing one value into an accumulator representing another value (or, to avoid the loss of the original separate values, creating two accumulators matching in value to the two to be added together, and "pouring" the contents of these two into a third, empty, accumulator). Subtraction could be similarly achieved: Suppose accumulator A represents the initial value, and accumulator B, the value to be subtracted from it. Successively removing increments of the contents of A -- or, to avoid loss of information of the initial value of A, the contents of A', a "copy" of A -- and "pouring" them into an initially empty accumulator C until C reaches the same fullness level as B (indicating the right amount has been removed) would result in A' representing the difference between the values represented by A and B.

Relationship of the Accumulator Mechanism to Later Numerical Knowledge

By virtue of yielding numerical representations in which magnitudinal information is inherently embodied, the accumulator mechanism, in conjunction with mental processes for operating over these representations, allows for the extraction of a rich body of numerical information. However, there are two sorts of limitations to the knowledge that could be obtained from the outputs of the accumulator model.

First, there will be practical limits on how the outputs of the mechanism can be manipulated. While an extensive body of numerical information is in principle accessible by virtue of the magnitudinal structure of the representations -- for example, multiplicative relationships, division (with remainder), whether a given value is even or odd, or prime -- access to these facts requires that the cognitive system have procedures for manipulating the representations in appropriate ways to obtain it. The kind of procedure required for determining the product of two values, for example, will be much more involved and complex than that for determining the sum of two values, requiring an additional accumulator to count the number of successive additions. A procedure for determining how many times one number goes into another will be more complex yet; one for computing an exponential value will be even more complex, and so on. It is an empirical question what kinds of processes comprise this system of numerical competence; the accumulator theory does not predict that the animal or infant must be able to appreciate *all* these kinds of numerical relationships.

Second, there are limits to what the accumulator is *in principle* capable of representing. One limitation is that it can only represent numbers *of specific individuals* (be they objects, sounds, events, or whatever). The body of mathematical knowledge that has developed through history and been transmitted culturally, however, operates as an abstract, logical system independent of the physical world. Somehow, the mind must make the abstraction, from conceiving of numerical properties of collections of entities, to conceiving of abstract numbers that are entities in themselves.

Another principled limitation to the accumulator is the kinds of mathematical entities and concepts it can give rise to. It is unlikely that the notion of infinity could result from this mechanism, as all physical processes are limited and there is presumably some point beyond which the accumulator cannot measure. Numbers other than the positive integers cannot be represented -- by its nature, the accumulator mechanism does not measure fractional values, negative values, imaginary values, and so on. Interestingly, representation of these kinds of numerical entities emerges very gradually and with much effort, both ontogenetically and culturally. Children's acquisition of fractions as numerical entities is extremely difficult, requiring an expansion beyond their initial conceptualization of numbers as 'things that represent numbers of discrete entities' (Gelman, Cohen, & Hartnett, 1989). More generally, this conceptual expansion can be seen in the historical development of mathematics, in numerous domains. The number 'zero', for example, was initially not part of the number system. It was introduced as a 'place-holder' symbol representing an absence of values in a given position in place-value numeral notation (so as to reliably distinguish, say, 307 from 37), and eventually attained status as a number in its own right by virtue of becoming embedded in the system, with rules for its numerical manipulation (Kline, 1972). The emergence of negative numbers, irrational numbers, complex numbers, and so on follow similarly complex and gradual progressions.

The earliest numerical thought involved the positive integers (Ifrah, 1985); the very values produced by the accumulator mechanism. Moreover, the earliest physical representations (and also, later on, the earliest written notations) for the numbers had a magnitudinal structure, in which a given number (of sheep, for example) was represented by a collection of that same number of pebbles or other 'counters' (Ifrah, 1985; Schmandt-Besserat, 1986, 1987). All these facts suggest that the positive integers are psychologically privileged numerical entities; and these are precisely the numerical entities that the accumulator model is capable of producing and representing.

The accumulator mechanism provides mental representations of numbers, and allows for manipulation of these representations in numerically meaningful ways. In this respect, the structure of these psychological foundations of numerical knowledge is similar to that of formal systems of mathematics. Such systems can be characterized by a body of mathematical entities (often defined by a set of axioms, e.g., the Peano axioms of arithmetic or Euclid's axioms of geometry) along with a set of operations performable upon these entities. To ultimately understand the role the initial numerical foundations supplied by the accumulator mechanism play in the development of further numerical knowledge, it will be critical to determine the numerical operations the accumulator mechanism can perform, as well as the 'axioms' that psychologically 'define' its outputs (see Wynn & Bloom, 1992), and to determine the ways in which these psychological foundations map onto the abstract system of arithmetic and the ways in which they differ.

The Accumulator Model and Linguistic Counting

Even within the domain of the positive integers, extensive conceptual development occurs. There are significant differences in the way the accumulator mechanism represents number, and the way in which number is represented in culturally transmitted, linguistic counting systems. In linguistic counting, an ordered list of number words are applied to the items to be counted in a one-to-one correspondence, and the final word used in the count represents the total number of items counted (Gelman & Gallistel, 1978). Number is not inherently represented in the symbols for the numbers (the number words); these obtain their numerical meaning by virtue of their positional relationships with each other -- the fifth word in the series represents the number *five*, and so on. The number words represent cardinalities through a system that is isomorphic to, but but distinct from, the the system of cardinals and their internal relations. For example, the number *six* is three units larger than, or twice as large as, the number *three,* while the linguistic symbol for *six* occurs three positions later in the number word list than the linguistic symbol for *three*, or twice as far along.

In order to master the linguistic counting system, then, children must learn the mapping between their own magnitudinal representations of number and the ordinal representations inherent in the linguistic counting system -- surely not a trivial process. A series of studies examining children's developing understanding of the linguistic counting system suggests that a mastery of linguistic counting is in fact a complex achievement, with an extensive period of learning.

In one experiment (Wynn, 1990b), 2-1/2- to 4-year-olds were asked to give a puppet from 1 to 6 toy animals from a pile (the "Give-a-number" task). If children understand how linguistic counting determines numerosity, they should be able to use counting to give the correct number. Children were also given a counting task, in which they were asked to count sets of items ranging in number from 2 to 6, and were asked, after counting, "how many" items there were.

In the Give-a-number task, there was a bimodal distribution in response strategy. Some children (the "Counters", who on average were about 3-1/2 years old), did use counting to solve the task, much as would an adult. They typically counted the items from the pile, stopping at the number word asked for, thus succeeding on the task in giving the right number. The other children (the "Grabbers", who were typically under 3-1/2 years of age), did not use counting to solve the task. They never counted items from the pile to give the correct number, even though they were quite good at performing the counting routine (for example, children who could count 6 items perfectly well were unable to give, say, 4 items from the pile). When asked for larger numbers, these children just grabbed and gave a random number of items. When asked for smaller numbers (3 and fewer), some children had apparently directly mapped some of these number words onto their correct numerosities (presumably as a result of an ability to subitize) and so could give the correct number just by looking, but not by using linguistic counting. A response made by a typical Grabber follows:

Experimenter: “Can you give Big Bird five animals?” (*The child grabs a handful that contains 3 animals and puts them in front of the puppet.*)

Experimenter: “Can you count and make sure there’s five?”

Child: (*Counting the three items perfectly*) “One, two, three -- that’s five!”

Experimenter: “No, I really don’t think that’s five. How can we fix it so there’s five?”

Child: (*Pauses, and then very carefully switches the positions of the 3 items*) “There. Now there’s five.”

Thus, the Grabbers appear not to appreciate just how linguistic counting determines the number of a collection of items.

A second analysis examined the relationship between children's performance in the Give-a-number task, and their responses to the "how many" questions following counting in the counting task. The "how-many" task has often been considered an indicator of whether a child understands that the last number word used in a count represents the number of items counted -- if a child understands this, it is sometimes argued, then when asked "how many" immediately after counting a collection of items, he or she should give the last number word used in the count as the response to the question. But there are reasons to doubt the validity of this task: Children might correctly answer a “how many” question without having the assumed knowledge, simply as a result of having observed adults modeling counting for them. Conversely, children might give a response other than the last number word used in the count even if they do understand the significance of the last word. Asking about the number of items immediately after the child has just counted them (and hence provided information as to their number) might cause the child to interpret the request as an indication that the count was incorrect, and to recount the items rather than stating the last word. For these reasons, whether or not a child gives the last word used in the count when asked “how many” may be not be a good indication of her grasp of the significance of the last number word in the count.

However, looking at *when* children give the last number word might be more valid. The last word used in a count is the correct response to "how many" only when the count has been executed correctly -- in an incorrect count in which the one-to-one correspondence of words to items has been violated, or in which the number words have been applied in an incorrect ordering, the last number word will not indicate how many items there are. If children understand how the counting system determines number, they might therefore be expected to give last-number-word responses less often after incorrect counts than after correct ones. If, however, they do not possess such an understanding, they should not differentiate correct from incorrect counts, but give last-number-word responses equally often after both kinds of counts. Accordingly, Grabbers' and Counters' responses to the "how many" questions were divided into those following correct counts by the child, and those following incorrect counts. Results showed that the Counters, but not the Grabbers, gave significantly more last-number-word responses after correct counts than they did after incorrect counts; Grabbers made no distinction between the two kinds of counts (see Figure 7), further strengthening the conclusion that they did not understand how the counting system represents number, despite their competence at performing the counting routine.[2]

INSERT FIGURE 7 ABOUT HERE

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[2]Gelman (1994) presents evidence suggesting that children's early understanding of linguistic counting may be significantly greater than these studies indicate. The experimenter first showed children a card with one item on it (e.g., a cat), asking children "What's on this card?" When children invariably responded "a cat", the experimenter would respond, "That's right; *one* cat." Children were then shown a card with 2 cats, again asked "What's on this card?" and, following their response, were asked "Can you show me?" The same questions were then asked using cards with larger numbers. Using this procedure, Gelman obtained a high percentage of young children correctly stating the number of items on the cards (at least for the numbers 2 and 3), and using counting in response to the experimenter's "Can you show me?" questions. Further research is needed to clarify children's understanding of the task, and explore the extent to which this finding affects the conclusions of the above discussion.

Intermediate Stages in the Developing Understanding of Number Words

The above conclusion leads to the question of the precise nature of children's understanding of both the purpose of the counting routine, and the meanings of the number words used in counting. In a longitudinal study (Wynn, 1992b), children's developing understanding of the meanings of number words was looked at in detail. Two aspects of knowledge of number words were examined: (1) whether children knew the semantic category that these words pick out, that is, that number words refer to numerosities; and (2) whether they knew exactly *which* numerosity each word picks out.

To address whether children knew that number words picked out numerosities, pairs of pictures were shown to children who already knew the meaning of the word "one". These children were asked to point out the picture depicting a certain number of objects. For example, children were shown one picture depicting a single blue fish and another depicting 4 yellow fish, and asked "Can you show me the four fish?" If they know that "four" refers to a specific number, they should (in accordance with the principle of contrast; see, e.g., Clark 1987; see also Markman 1989) infer that it does not refer to the number *one* since they already have a word for the numerosity *one*. They should thus choose the correct picture by a process of elimination. However, if they do not know that "four" is a number word, they will not know to contrast it with "one", and should respond as they do when asked a nonsense question, such as "can you show me the *blicket* fish?" (When children were asked this, they were equally likely to point to either picture.)

To address whether children knew the precise number a number word picks out, they were shown pairs of pictures, one of which contained a given number of items, the other of which contained that number plus one. For example, to test whether children knew the precise meaning of the word "four", they were shown a picture of 4 green dogs and one of 5 red dogs, and asked "Can you show me the four dogs?" on some trials and "Can you show me the five dogs?" on others. (Again, they were asked nonsense questions as well, such as, “Can you show me the *blicket* dogs?”, and chose each picture equally often on these questions).

Even the youngest children (2-1/2-year-olds) succeeded when one of the pictures in the pair contained a single item. They correctly pointed out the number asked for 94% of the time, showing a clear understanding that the number words pick out numerosities. However, despite this early knowledge, it took children nearly a full additional year to learn the general pattern for which words refer to which numerosities -- i.e., that successive words in the number word list refer to successively higher numbers. The same pattern of results was found as with the Give-a-number task above: When asked to distinguish small numbers (2 from 3 and, sometimes, 3 from 4) some children had directly associated that number with its correct number word. They thus succeeded at these smaller numbers, but never by using counting to determine the answer. When presented with larger numbers (e.g., pictures of 4 and 5 dogs), the only children who succeeded were children who counted the items in the pictures (these children tended to be 3-1/2 years of age or older). These children succeeded on all the numbers they were asked to identify, and tended to use counting even when asked for the smaller numbers, showing that once they understand that linguistic counting is a method for determining number, they can then extend their use of this strategy even to situations for which they already have another strategy available.

These studies show that there appears to be a lengthy period of time during which young children know how to apply the counting routine, and know that these words each refer to a specific numerosity, but do not know the system behind which numerosity each word picks out -- the ordinal system by which the list of counting words represents numerosity. This protracted period of learning is consistent with the accumulator model, which posits initial numerical representations very different in form from those employed by the linguistic counting system. On the accumulator model, what children must do in order to learn the linguistic counting system is to map its ordinal representations of number onto children's own initial, magnitudinal representations of number, by making a kind of analogical mapping: *Larger* numbers are represented by symbols occurring *later* in the number word list, *smaller* numbers by symbols occurring *earlier* in the list.

This raises a number of intriguing questions. If the linguistic counting system represents number in a very different way from the initial accumulator representations, then its acquisition should have a major influence on numerical thought. To acquire a new representational system, one which exploits different properties of the entities to be represented, is to acquire a new tool with which to think and reason about the entities under consideration. The counting system embodies a wealth of information about number not available from the accumulator mechanism -- for example, information about the iterative structure of the number naming system, which may in turn provide information about numerical infinity (Bloom, 1994a). An understanding arrived at in acquiring the linguistic counting system that may be particularly important is that numbers can be represented by arbitrary symbols, whose structures are independent of the values they represent. This allows the possibility of representing variables, which opens the door to an entirely new realm of mathematical concepts, representations, and entities, and is surely a major achievement in the development of mathematical reasoning, both in the individual and in the history of mathematics. This achievement would be impossible within any system in which the structures of the symbols representing numbers are inextricably dependent on the values of those numbers, such as the accumulator mechanism.

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Figure Legends

Figure 1. Six-month-olds' looking to puppet following novel versus habituated number of jumps (Experiment 1 of Wynn, 1995: motionless pause between jumps)

Figure 2. Six-month-olds' looking to puppet following novel versus habituated number of jumps (Experiment 2 of Wynn, 1995: puppet in constant motion between jumps)

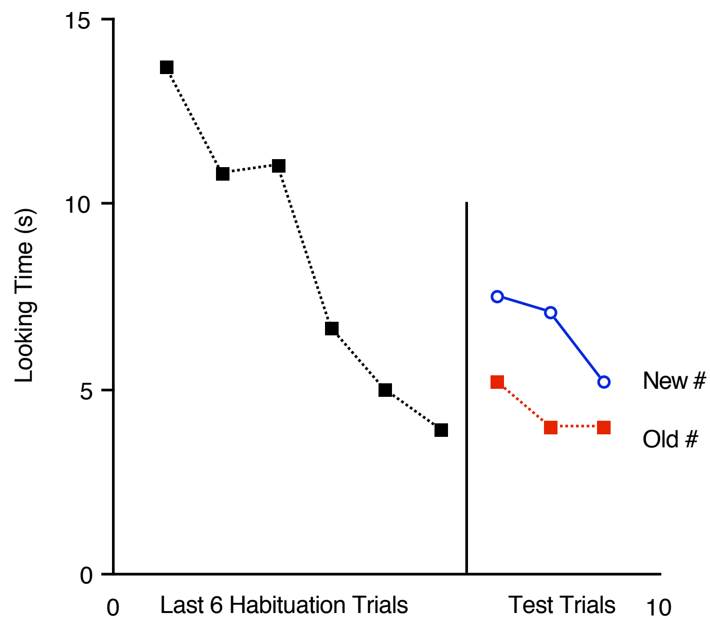
Figure 3. Sequence of events shown infants in Wynn (1992a), Experiments 1 and 2 (courtesy of *Nature*)

Figure 4. Five-month-olds' looking to 1 versus 2 objects, following either "1+1" or "2-1" sequences of events

Figure 5. Five-month-olds' looking to 2 versus 3 objects, following a "1+1" sequence of events

Figure 6. Five-month-olds' looking to either 2 objects or 3 objects, following "2+1" versus "3-1" sequences of events

Figure 7. Percentage of Grabbers' versus Counters' responses to "how many" question following correct and incorrect counts in which they gave the last number word used in the count



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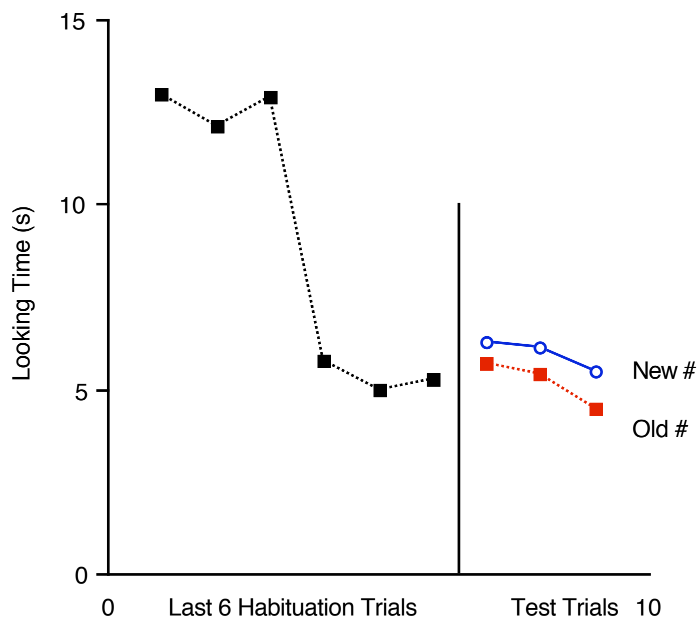
10
5
0
Looking Time (s)
0
Last 6 Habituation Trials
Test Trials
10
New #
Old #



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Looking Time (s)
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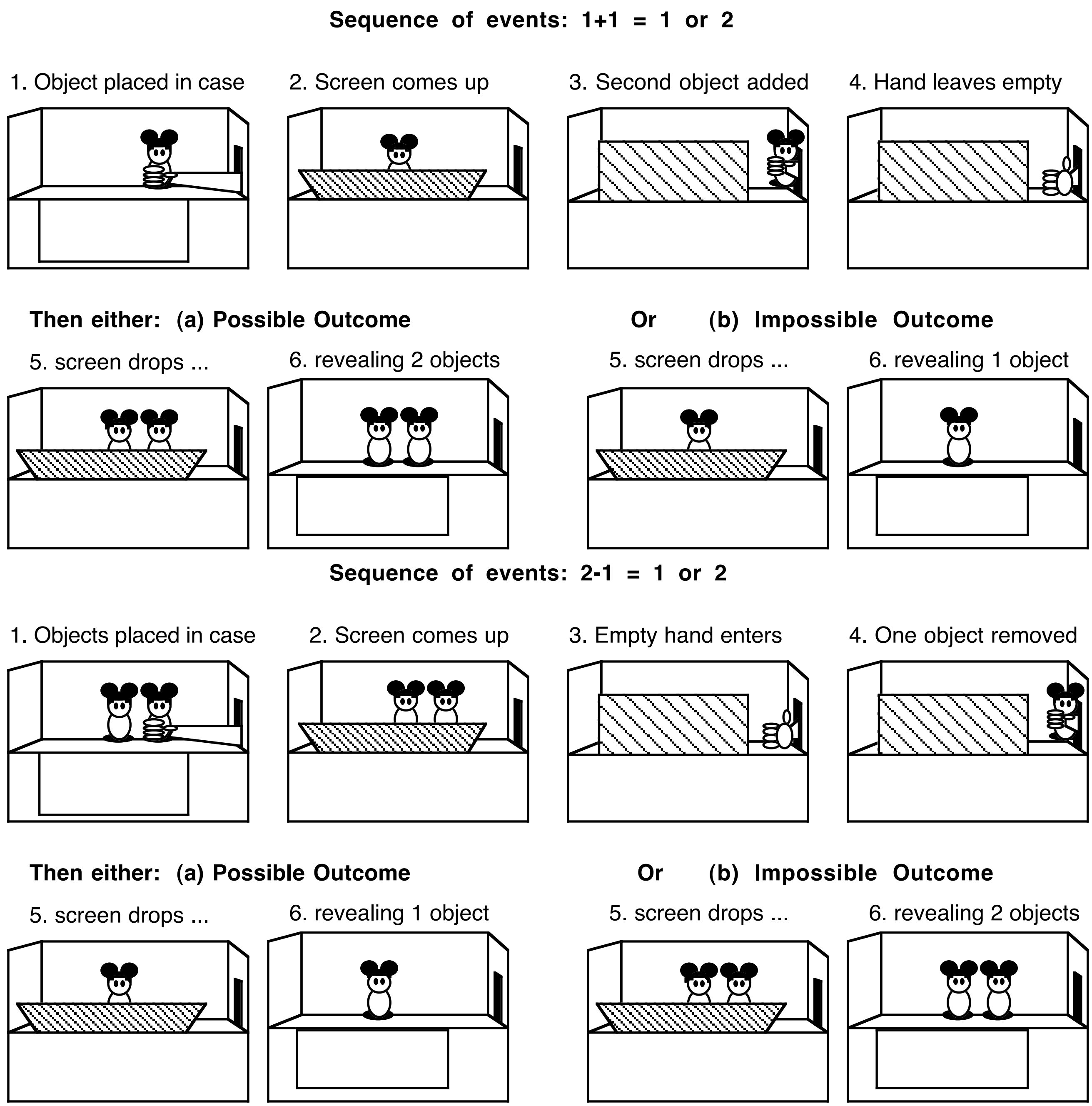
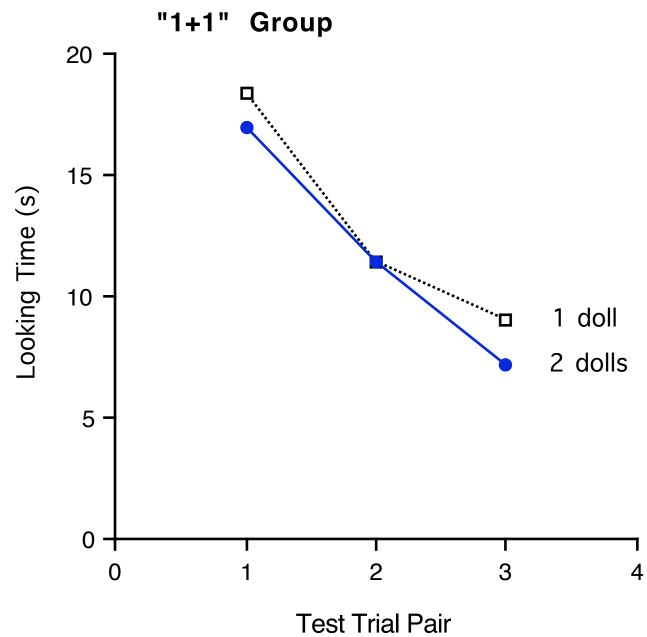
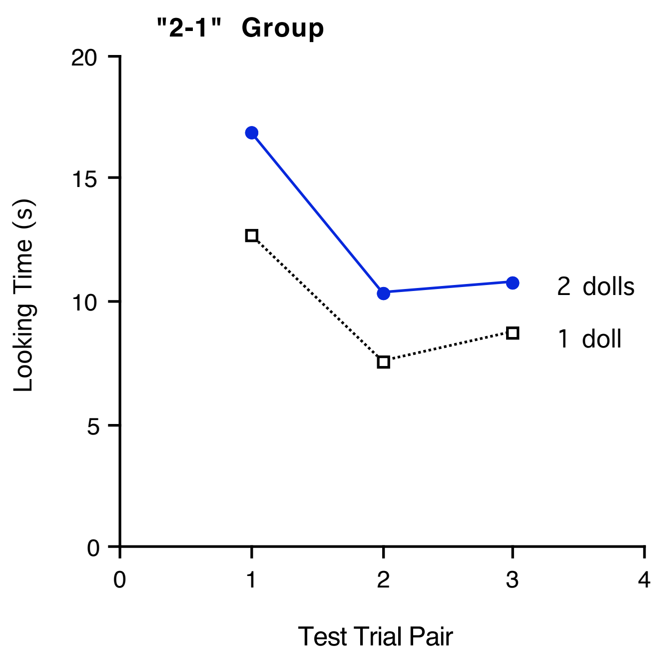
Last 6 Habituation Trials
Test Trials
10
New #
Old #


Sequence of events: 1+1 = 1 or 2
1. Object placed in case
2. Screen comes up
3. Second object added
4. Hand leaves empty
Then either: (a) Possible Outcome
5. screen drops ...
6. revealing 2 objects
Or (b) Impossible Outcome
5. screen drops ...
6. revealing 1 object
Sequence of events: 2-1 = 1 or 2
1. Objects placed in case
2. Screen comes up
3. Empty hand enters
4. One object removed
Then either: (a) Possible Outcome
5. screen drops ...
6. revealing 1 object
Or (b) Impossible Outcome
5. screen drops ...
6. revealing 2 objects


"1+1" Group
Looking Time (s)
20
15
10
5
0
1 doll
2 dolls
0
1
2
3
4
Test Trial Pair


"2-1" Group
Looking Time (s)
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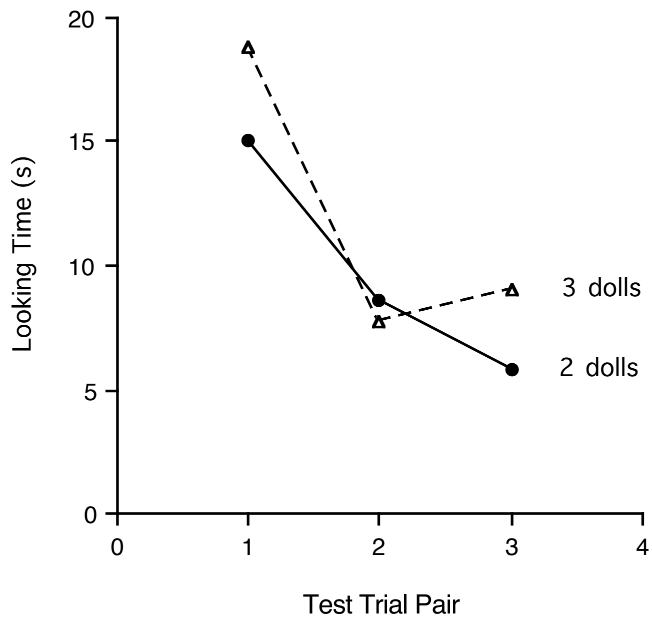
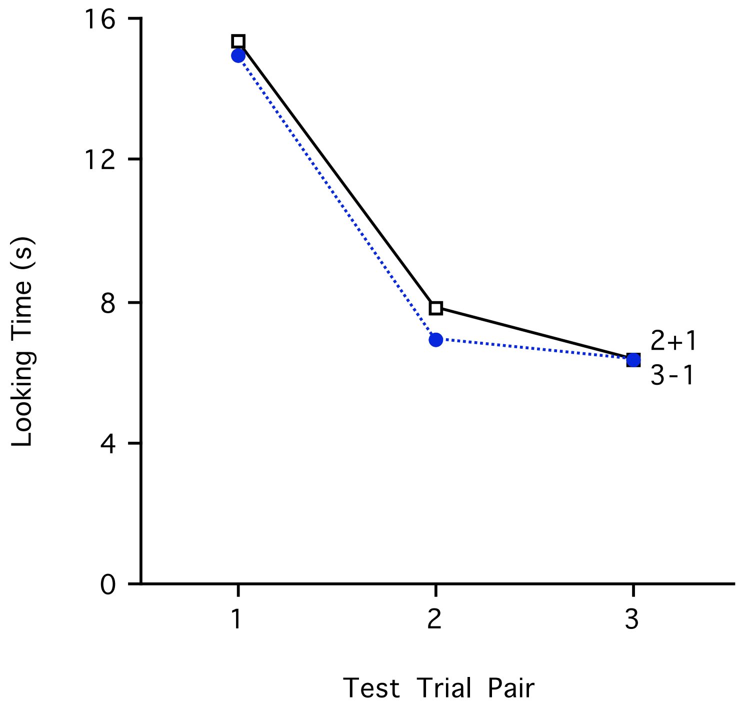
15
10
5
0
2 dolls
1 doll
0
1
2
3
4
Test Trial Pair

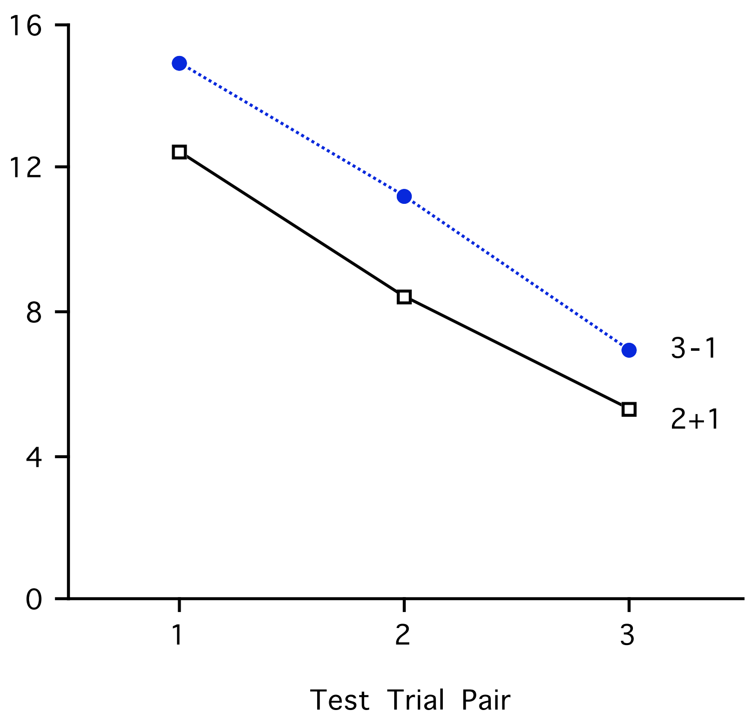


20
15
10
5
0
Looking Time (s)
3 dolls
2 dolls
0
1
2
3
4
Test Trial Pair

Outcome of Two Dolls



Outcome of Three Dolls



Mean % Cardinality Responses Following Correct vs Incorrect Counts