

# Fast hash-based additive accumulators

[DRAFT]

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## Abstract

Cryptographic accumulators allow to succinctly represent a large set of values by providing a membership witness that can be used to verify that the element was indeed part of a committed set. Accumulators could be used in a public blockchain to create accountable protocols, in which a party could prove to a smart contract that certain events have happened without having to store an expensive audit log. In this work we introduce the first append-only hash-based accumulators with sub-logarithmic insertion cost and poly-logarithmic witness size.

## 1 Introduction

Cryptographic accumulators are short commitments to a set  $S$  in a way that allows one to produce compact proofs of membership that can be verified efficiently by anyone with the knowledge of the accumulator's value.

Introduced in [BDM93] with an application to timestamping, accumulators find a wide range of applications, for example in accountable certificate management [NN98, BLL00], authenticated dictionaries [GTH02], anonymous credentials [CL02], computations on authenticated data [ABC<sup>+</sup>15], anonymous E-cash [STS99, MGGR13, SALY17], data outsourcing [Sla12], updateable signatures [CJ10, PS14], and decentralized bulletin boards [FVY14, GGM14].

Accumulators find natural applications in blockchains. They are among of the building blocks of Zerocoin [MGGR13], a protocol for anonymous E-cash. Cryptocurrencies like Bitcoin [Nak08] need all network nodes to reach

consensus on the set of *unspent transaction outputs* (UTXOs). The size of the UTXO set has grown to several gigabytes over time, which increases the cost of running a node, therefore affecting the decentralization of such networks. Similar concerns equally apply to cryptocurrencies in the account model like Ethereum. [BBF19] builds constant-sized accumulators using groups of unknown order, and uses them to create *stateless blockchains* with lightweight nodes. While this construction is theoretically the best possible and it has other useful properties like batching, it has the downside of requiring additional cryptographic assumptions compared to hash-based accumulators. Moreover, hash functions are in practice orders of magnitude faster than operations in number-theoretic groups. In fact, [Dry19] builds on the dynamic hash-based accumulators of [RY16] to show how to create efficient, hash-based dynamic accumulators for the UTXO set of Bitcoin.

## 1.1 Append-only audit logs

In some protocols, there is an asymmetry in the operational costs of updates, proof generations and proof verification. For example, in Pisa [MBB<sup>+</sup>18], an audit log of the actions executed by the operator is used in order to later be able to prove that such actions were indeed executed. Thus, if the operator was hired for a task and it did not complete it, the user can issue a challenge, and the operator will lose it and be slashed; vice versa, if the user issues a challenge despite the operator has done the task, then there will be an entry in the audit log that the operator can use to win the challenge.

The downside of the approach above is that the size of the state that grows linearly with the number of tasks. This can be solved by using an accumulator.

Note that in the above protocol, it is not expected to actually have to generate or verify proofs frequently. In fact, if the operator did complete the task, a rational user will *never* issue a challenge that will lose. Thus, while it is important that the costs of are small enough that a proof *could* be created and verified, it is game-theoretically not expected to happen. On the contrary, the accumulator needs to be updated with a new item every time a new task is performed. This justifies the search for accumulators whose update cost is as small as possible, even at the cost of slightly larger proof sizes.

We remark that there is no requirement of ever *deleting* elements that were added to the accumulator. That is, using the nomenclature in [BCD<sup>+</sup>17],

an *additive* accumulator suffices, rather than a fully dynamic accumulator. It is reasonable to expect that one can obtain an improvement on proof size and update costs by reducing the set of allowed operations on the accumulator, and this is the topic of this paper.

## 1.2 Our contribution

In this work, we propose a hash-based *additive* accumulator with extremely efficient element insertion time, while retaining poly-logarithmic proof size and verification times. Unlike static accumulators, our constructions allow to add new elements to the committed set, but fall short of being fully dynamic in that we do not allow to remove elements.

Our first construction has a  $O(1)$  insertion time (notably, only one hash and the counter need to be written at each insertion), while the proof size is  $O(\log^2 n)$ , where  $n$  is the number of elements added to  $S$  from the beginning.

By storing the state of our first construction in a Merkle tree, we obtain an accumulator with  $O(\log \log n)$  insertion time, and proofs of size  $O(\log n \log \log n)$ . In both cases, the space occupation for Add is  $O(\log n)$ , while the accumulator is a single hash.

To the best of our knowledge, previous hash-based accumulator constructions are based on Merkle trees ([Mer80]) or forests of Merkle trees ([RY16], [Dry19]), and have  $O(\log n)$  insertion time and proof size.

## 2 Definitions

We assume that  $H$  is a collision-resistant hash function, and we denote with  $\|$  the concatenation operator.

For any positive integer  $n$ , we define  $d(n)$  as the largest power of 2 that divides  $n$ . For example,  $d(17) = 1$ ,  $d(6) = 2$  and  $d(24) = 8$ . When  $n$  is expressed in binary,  $d(n)$  is the number obtained by zeroing all 1 digits, except the least significant one, for example  $d(88) = d(1011000_2) = 1000_2 = 8$ .

For a positive integer  $n$ , we further define  $zeros(n) = \log_2(d(n))$  the number of trailing zeros of  $n$ , and we define the *predecessor*  $pred(n) := n - d(n)$ . It is easy to see that  $pred(n)$  is the number obtained by zeroing the rightmost 1 bit in the binary representation of  $n$ . For example  $pred(88) = pred(1011000_2) = 1010000_2 = 80$ .

For convenience, we also define  $pred(0) = zeros(0) = 0$ .

We also define  $pred_1(n) := pred(n)$  and, for  $t \geq 2$ , we define  $pred_t(n) := pred(pred_{t-1}(n))$ ; that is,  $pred_t(n)$  is the result obtained by starting from  $n$  and applying  $t$  times the transform  $n \mapsto pred(n)$ .

### 3 Intuition

In this section we present the main idea behind our new construction for an additive accumulator.

Let  $R_k$  be the value of the accumulator after the  $k$ -th element  $x_k$  is added. We consider schemes where  $R_k$  is defined as a commitment to the value of  $x_k$ , together with the value of some subset of previous accumulators  $\{R_l\}_{l \in P_k}$ , where  $P_k \subseteq [1, k - 1]$ . Therefore, once  $R_k$  is known, a witness for some element  $x_j$  with  $j < k$  can be produced by opening the commitment for  $R_k$  itself to prove the values of  $x_k$  and  $R_{k'}$  for an appropriately chosen  $k' \in P_k$ , and concatenating it with the witness for  $x_j$  starting from  $R_{k'}$ .

In the graph that has a node for each  $R_j$ , and a directed arch from  $R_j$  to  $R_i$  if  $i \in P_j$ , building a witness for some element  $x_j$  amounts to finding a path from the current accumulator's node  $R_k$  to the node corresponding to  $R_j$ , and revealing all that is necessary to verify the chain of commitments; by rebuilding the same path, one can verify such a witness. Of course, it is best to find a path that is as short as possible.

The most trivial such scheme is obtained by setting  $P_k := \{k - 1\}$  for each  $k > 1$ , obtaining a chain of commitments. This is not an interesting accumulator, though, as the length of the witness is  $O(k - j)$ .

A more interesting construction can be obtained by choosing

$$P_k := \{R_{k-1}, R_{k-2}, R_{k-4}, \dots, R_{k-2^i} \dots\}$$

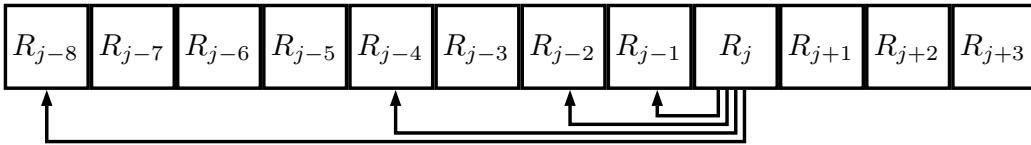


Figure 1: A simple accumulator where each new value  $R_j$  commits to previous accumulators at geometrically increasing distance. Each arrow represents one of the values of previous accumulators that  $R_j$  commits to.

As  $|P_k| = \lceil \log k \rceil$ , it is straightforward to obtain a construction with poly-logarithmic witness size, by always revealing  $R_{k-2^i}$  where  $i$  is chosen so that  $k - 2^i$  is the smallest number that is bigger than or equal to the target  $j$ . This construction presents a major drawback: the size of the accumulator's state grows linearly in  $k$ , and any party that wishes to verify that updates are correct needs to replicate the same state. In the following sections we present our constructions where we address this problem by carefully choosing  $P_k$ .

A crucial idea in all our construction is the following observation: for any fixed integer  $d > 1$ , the largest multiple of  $d$  not bigger than  $k + 1$  is equal to the highest multiple of  $d$  not larger than  $k$ , *unless*  $k + 1$  itself is a multiple of  $d$ .

Therefore, we use as 'hook' to the past history of the accumulator the largest  $k' \leq k$  that is divisible by  $2^i$  but not divisible by  $2^{i+1}$ . We do this for each  $i$  such that  $2^i \leq k$ , obtaining a set of  $\lceil \log n \rceil$  hashes that are stored in the state of the accumulator. Crucially, when a new element  $x_{k+1}$  is added, only one of these hooks need to be updated, corresponding to the highest power of two that divides  $k + 1$ .

## 4 First scheme: smart back-links

In our first construction, we choose  $P_k = \{k - 1, \text{pred}(k)\}$ , using the notation of Section 1<sup>1</sup>.

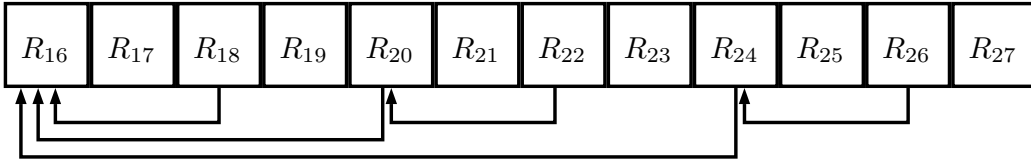


Figure 2: Representation of the commitments of our construction for the accumulator's values  $R_{16}$  to  $R_{27}$ . Each  $R_i$  also commits to the  $R_{i-1}$ , but the arrow is omitted for simplicity; for even values of  $i$ ,  $R_i$  also commits to  $R_{\text{pred}(i)}$ .

The public state of our accumulator contains a counter  $k$ , initially set to 0, and an array  $S$ . We assume that  $S[i] = 0$  by default for every integer

<sup>1</sup>If  $k$  is odd,  $\text{pred}(k) = k - 1$ , hence it is redundant to commit to both values; we ignore such optimizations for the sake of simplicity.

$i$ , until  $S[i]$  is assigned a different value. The value of  $k$  is the number of elements that were added to the accumulator from the beginning.

At any point in time, we maintain the following invariant:  $S[i]$  for a non-negative  $i$  is the value of the accumulator when the largest  $k' \leq k$  was inserted with the property that  $k'$  is divisible by  $2^i$ , but not divisible by  $2^{i+1}$ ;  $S[i] = 0$  if no such number exists.

In particular, the current accumulator  $R_k$  after  $x_k$  was added is always the value stored in  $S[\text{zeros}(k)]$ ; we call  $R_k$  the value of the accumulator after the  $k$ -th element is added. In order to simplify the notation, we also define  $R_0 = \text{NIL}$ .

Only the first  $O(\lceil \log k \rceil)$  elements of  $S$  can be non-zero, so no other element needs to be stored as part of the public state of the accumulator.

The accumulator's operator (or anyone that wishes to produce witnesses) needs to also store the value  $R_k$  of the accumulators for each  $k$ .

**Element insertion** In order to add the  $k$ -th element  $x_k$ , the accumulator's operator computes  $R_k$  as produces a new commitment (by hashing) to the concatenation of  $H(x_k)$ ,  $R_{k-1}$  and  $R_{\text{pred}(k)}$ .

The resulting hash is stored in  $S[\text{zeros}(k)]$ , which is the new accumulator's root.

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**Algorithm 1:** Inserting element  $x$

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ADD( $x$ )
 $k \leftarrow k + 1$ 
 $prev \leftarrow S[\text{zeros}(k - 1)]$ 
 $parent \leftarrow S[\text{zeros}(\text{pred}(k))]$ 
 $S[\text{zeros}(k)] \leftarrow H(x || prev || parent)$ 

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Once the  $k$ -th element  $x$  is inserted in the accumulator, we assume that the operator stores  $R_k$  and  $x_k$  forever if it needs to be able to produce proofs of membership. Note that storing  $R_k$  is only needed to produce proofs of membership.

Observe that the value of  $prev$  in the algorithm above is equal to  $R_{k-1}$ , while the value of  $parent$  equals  $R_{\text{pred}(k)}$ . Therefore it is always the case that:

$$R_k = H(x_k || R_{k-1} || R_{\text{pred}(k)})$$

**Witness generation** Given the value of  $R_i$  for some  $i$ , the operator can generate a membership witness of the value of  $x_j$ , for any  $0 < j \leq i$ .

We define  $\pi_i = (x_i, R_{i-1}, R_{pred(i)})$ . Note that  $\pi$  is the opening of the commitment  $R_k$ .

If  $j = i$ , the witness consists of the values of  $R_{i-1}$  and  $R_{pred(i)}$ , necessary to open the commitment for the accumulator  $R_i$ .

If  $j < i$ , then we recursively generate the witness starting from either  $R_{i-1}$  or  $R_{pred(i)}$ , whichever is "most convenient". That is, we always use the witness for  $R_{pred(i)}$ , unless  $pred(i) < j$ ; in that case, we concatenate  $\pi_i$  with the witness for  $R_{i-1}$ .

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**Algorithm 2:** Generating a witness for  $x_j$  starting from  $i$

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MEMWITCREATE( $i, j$ )
   $\pi \leftarrow (x_i, R_{i-1}, R_{pred(i)})$ 
  if  $i = j$  then
    return  $\pi$ 
  else
    if  $pred(i) < j$  then return  $\pi \parallel \text{MEMWITCREATE}(i - 1, j)$ 
    else return  $\pi \parallel \text{MEMWITCREATE}(pred(i), j)$ 
  end

```

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Remark: the above algorithm generates the witness for element  $x_j$ , starting from the accumulator at time  $i$ . The witness for element  $x_j$  for the current accumulator is therefore generated by calling  $\text{MEMWITCREATE}(k, j)$ .

**Witness verification** Given a witness  $w$  for the value of  $x_j$  given the value of  $R_i$ , for some positive  $j \leq i$ , the verifier first checks that the hash of the concatenation of the first three elements of  $w$  is indeed  $R_i$ . Let  $w'$  be equal to  $w$  after removing the first 3 elements of  $w$ , and let  $x, P, Q$  the first three elements of  $w$ , respectively. Then:

- If  $j = i$ , it checks that the first element of  $w$  is indeed  $x_j$ .
- If  $j < i$  and  $pred(i) < j$ , it recursively verifies that  $w'$  is a valid proof of the value of  $x_j$  given that the value of  $R_{i-1}$  is  $P$ .
- Otherwise, if  $j < i$  and  $pred(i) \geq j$ , it recursively verifies that  $w'$  is a valid proof of the value of  $x_j$  given that the value of  $R_{pred(i)}$  is  $Q$ .

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**Algorithm 3:** Verifying that  $w$  is a valid proof that  $x_j = x$ , given  $R_i$ .

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VERMEM( $R_i, i, j, w, x$ )

Parse  $w$  as  $x', P, Q, w'$ , where  $x', P, Q$  are the first three elements of  $w$ , and  $w'$  is the rest of the list.

**if**  $H(x' || P || Q) \neq R_i$  **then**  
    **return** 0

**end**

**if**  $i = j$  **then**  
    **return** 1 if  $x = x'$  else **return** 0

**else**

**if**  $\text{pred}(i) < j$  **then** **return** VERMEM( $P, i - 1, j, w', x$ )  
    **else** **return** VERMEM( $Q, \text{pred}(i), j, w', x$ )

**end**

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Similarly to the witness generation function, the VERMEM verifies that  $w$  is a valid proof that the given element  $x$  is indeed element the  $j$ -th element added to the accumulator, given that the accumulator's value for some  $i \geq j$  is  $R_i$ . To verify a proof against the current accumulator after  $k$  elements have been added, one calls the VERMEM function with first parameters  $R_k$  and  $k$ , respectively.

**Witness update** As there is no trapdoor and the execution of the ADD function is public, any party that was observing the accumulator can generate the witness for a specific  $x_k$ ; in that case, it does not need more than a number of elements proportional to the size of the proof itself.

With a more careful algorithm, the running time of the witness update can be slightly improved. In fact, the old witness and the new witness are identical, except for a prefix of length at most  $O(\log n)$  that is replaced from the old witness.

## 4.1 Analysis

**Theorem 1.** *The construction defined above is an additive accumulator with  $O(1)$  insertion cost, size of the accumulator state  $O(\log n)$ , and length of the proof  $O(\log^2 n)$ . The cost of a witness update is  $O(\log n)$ .*

**Remark.** *More precisely, the size of the proof for element  $j$  when  $k$  elements*



have been added to the accumulator is  $O(\log^2(k - j))$ .

## 5 Second scheme: Merkleized back-links

In this section we sketch the modifications to the above scheme to obtain a different trade-off in running time; namely, we modify the commitment for the  $k$ -th element to include the roots for  $O(\log n)$  previous elements instead of one. This will allow to obtain quasi-logarithmic proof size, at the cost of a slightly worse insertion cost of  $O(\log \log n)$ .

The main observation is that since  $\text{pred}(n)$  is the number obtained by zero-ing the least significant 1 of  $n$ , in the above construction we end up revealing a sequence of past commitments corresponding to the sequence  $\text{pred}(k), \text{pred}(\text{pred}(k)), \dots$  and so on, until we reach a number  $k'$  such that  $\text{pred}(k')$  is too small; in that case we reveal the commitment for  $k' - 1$  instead. Moreover, observe that all the commitments in the sequence  $R_k, R_{\text{pred}(k)}, \dots, R_{k'}$  are indeed available as elements of the array  $S$ .

Based on these observations, we modify the above construction as follows:

- The accumulator's operator stores and maintains in its state the entire Merkle tree that has the elements of the array  $S$  as leafs; let  $M_k$  be the value in the root of this Merkle tree.
- We define  $R_k := H(x_k || R_{k-1} || M_k)$ , in order to commit to both the element  $x_k$  and the state of the whole array  $S$ . We assume  $M_0 := NIL$

As the size of the array  $S$  is  $\lfloor \log k \rfloor$ , it follows that it is possible to reveal a specific leaf of the Merkle tree by revealing  $O(\log \log k)$  hashes. This allows to modify the witness generation (and similarly the witness verification) algorithm as follows: when  $\text{pred}(i) \geq j$ , instead of recurring over  $\text{pred}(i)$ , we recur over  $\text{pred}_t(i)$  for the largest  $t$  such that  $\text{pred}_t(i) \geq j$ .

The cost of updating the Merkle tree when an element of the array  $S$  is modified (or appended) is  $O(\log \log k)$  as well, since only the hashes from a leaf to the root need to be recomputed by the accumulator operator.

**Theorem 2.** *There is an additive accumulator that has with insertion cost  $O(\log \log n)$  and proof size  $O(\log n \log \log n)$  for an accumulator with  $n$  elements. The cost of a witness update is  $O(\log \log n)$ .*

## 6 Remarks and future extensions

The following ideas are still underdeveloped.

- In some applications, it could be reasonable to limit proofs to latest  $t$  elements most recently added, for some constant  $t$ ; in this case, the computational and size bounds are polylogarithmic in  $t$ .
- Both the above schemes can be modified to support simultaneous insertion of a batch of multiple elements at a cost significantly lower than what would be paid by inserting the elements one by one.

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